Notes on the Newman-Penrose formalism

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Abstract

These are notes on the Newman-Penrose formalism, which I wrote up while working on [RLGP21]. I refer to a Mathematica note several times, which can be accessed at:

`https://github.com/JLRipley314/2nd-order-teuk-derivations`.

Nothing in here is new (these notes mostly follow Chapter 1 of Chandrasekhar's book on black hole perturbation theory [Cha02], except for the section on the GHP formalism). Please let me know if you find any typos/errors!

Contents

Chapter 1

Setup of basic formalism: Einstein equations and null frames

1.1 Notation and summary of (sub)Riemannian geometry and the Einstein equations

We mostly follow the conventions (and order of presentation) of [Cha02], except when otherwise noted. Spacetime indices will be denoted with lower case latin letters. The comma will denote partial differentiation and the semicolon will denote covariant differentiation, although I'll also use ∂_i and ∇_i as well. I will bold font tensors when I do not give indices.

The metric signature is $+ - --$.

The Ricci identity is

$$[\nabla_k, \nabla_l] V^j = R^j{}_{ikl} V^i + T^n{}_{kl} \nabla_n V^j, \tag{1.1}$$

where $T^n{}_{kl}$ is the torsion. We set the torsion to be zero. From the Ricci identity, the Riemann tensor is

$$R^j{}_{lnm} = \Gamma^j_{lm,n} - \Gamma^j_{ln,m} + \Gamma^j_{kn}\Gamma^k_{lm} - \Gamma^j_{km}\Gamma^k_{ln}, \tag{1.2}$$

where the Γ^k_{ij} are the metric compatible (Christoffel) connection coefficients. In a coordinate basis we may write

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{jl,k} + g_{kl,j} - g_{jk,l} \right), \tag{1.3}$$

The Ricci tensor and Ricci scalar are

$$R_{ij} \equiv R^k{}_{ikj}, \qquad R \equiv g^{ij} R_{ij}. \tag{1.4}$$

The Jacobi identity for the commutator of two derivatives along with the Ricci identity gives us the following cyclic identity for the Riemann tensor

$$R^j{}_{lkm} + R^j{}_{klm} + R^j{}_{mkl} = 0. \tag{1.5}$$

In a coordinate basis, we can write the Bianchi identities as

$$\nabla_r R^j{}_{lpq} + \nabla_p R^j{}_{lqr} + \nabla_q R^j{}_{lrp} = 0. \tag{1.6}$$

The Weyl tensor is

$$C_{ijkl} \equiv R_{ijkl} - \frac{1}{n-2}\left(g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk} + g_{jl}R_{ik}\right) + \frac{1}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)R, \tag{1.7}$$

where n is the dimension of the manifold; we will always work with $n = 4$. It obeys the same algebraic symmetries as the Riemann tensor, and it is tracefree on all indices.

The Einstein equations are

$$G_{ij} \equiv R_{ij} - \frac{1}{2}g_{ij}R = T_{ij}, \tag{1.8}$$

where G_{ij} is the Einstein tensor, and T_{ij} is the stress-energy tensor. From the Bianchi identities, the Einstein tensor is divergence free; $\nabla^i G_{ij} = 0$.

1.2 Tetrad formalism

1.2.1 Basic setup

At every spacetime point we set up a basis of four vectors

$$e^i_{(a)}, \tag{1.9}$$

where a is the basis index, and i is the coordinate index. When we refer to a particular component, e.g. $i = 1$, we write $e^1_{(a)}$, and if $a = 1$ we write $e^i_{(1)}$. We raise/lower the spacetime indices with the metric/inverse metric g_{ij}/g^{ij}, respectively. We raise/lower the basis index with a constant, symmetric matrix and its inverse $\eta_{(a)(b)}/\eta^{(a)(b)}$, respectively. We call $\eta_{(a)(b)}$ the internal metric. We define $e^{(a)}_i$, etc. so that

$$e^i_{(a)}e^{(a)}_j = \delta^i_j, \qquad e^i_{(a)}e^{(b)}_i = \delta^{(b)}_{(a)}, \qquad e^i_{(a)}e_{(b)i} = \eta_{(a)(b)}, \qquad e^{(a)}_i e_{(a)j} = g_{ij}, \tag{1.10}$$

where δ is the Kronecker delta. For any given tensor we have the notation, e.g. $V^{(a)} = e^{(a)}_i V^i$.

1.2.2 Intrinsic derivative and Ricci rotation coefficients

We define the directional derivatives

$$\mathbf{e}_{(a)} \equiv e^i_{(a)}\nabla_i. \tag{1.11}$$

Recall for a coordinate basis we have $e^i_{(a)}$ "$=$" $\delta^i_{(a)}$. The notation for scalars will be

$$\phi_{,(a)} \equiv \partial_{(a)}\phi \equiv e^i_{(a)}\partial_i\phi. \tag{1.12}$$

The notation for tensors is

$$\begin{aligned}\partial_{(b)}V^{(a)} \equiv& e^j_{(b)}\nabla_j\left(e^{(a)}_i V^i\right)\\ =& e^j_{(b)}V^i\nabla_j e^{(a)}_i + e^j_{(b)}e^{(a)}_i\nabla_j V^i.\end{aligned} \tag{1.13}$$

We define the intrinsic derivative

$$V^{(a)}{}_{|(b)} \equiv \nabla_{(b)}V^{(a)} \equiv e^j_{(b)}e^{(a)}_i\nabla_j V^i, \tag{1.14}$$

and the Ricci rotation coefficients

$$\gamma_{(c)(a)(b)} \equiv e^k_{(c)}\left(\nabla_i e_{(a)k}\right)e^i_{(b)}, \tag{1.15}$$

$$\implies \nabla_m e_{n(a)} = \gamma_{(c)(a)(b)}e^{(c)}_n e^{(b)}_m. \tag{1.16}$$

As $\partial_i\eta_{(a)(b)} = 0$, we have

$$\gamma_{(c)(a)(b)} = -\gamma_{(a)(c)(b)}. \tag{1.17}$$

We next define

$$\lambda_{(a)(b)(c)} \equiv \left(\partial_j e_{(b)i} - \partial_i e_{(b)j}\right)e^i_{(a)}e^j_{(c)}. \tag{1.18}$$

With symmetric (e.g. Christoffel) connections, we may replace the partial derivatives with covariant derivatives, and we then invert the above relationship to write the Ricci rotation coefficients in terms of the $\lambda_{(a)(b)(c)}$:

$$\gamma_{(a)(b)(c)} = \frac{1}{2}\left(\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}\right). \tag{1.19}$$

1.2.3 Structure constants in terms of Ricci rotation coefficients

We mention the structure constants, which are defined to be

$$\left[e^i_{(a)}\nabla_i, e^j_{(b)}\nabla_j\right] \equiv C^{(c)}{}_{(a)(b)}e^k_{(c)}\nabla_k. \tag{1.20}$$

Acting this on a scalar function, we see that

$$C^{(c)}{}_{(a)(b)} = \gamma^{(c)}{}_{(b)(a)} - \gamma^{(c)}{}_{(a)(b)}. \tag{1.21}$$

1.2.4 Riemann tensor in terms of Ricci rotation coefficients

We have

$$
\begin{aligned}
R_{(a)(b)(c)(d)} =& R_{pqrs} e^p_{(a)} e^q_{(b)} e^r_{(c)} e^s_{(d)} \\
=& \left(\nabla_s \left(\nabla_r e_{q(a)}\right) - \nabla_r \left(\nabla_s e_{q(a)}\right)\right) e^q_{(b)} e^r_{(c)} e^s_{(d)} \\
=& \gamma_{(a)(b)(d)|(c)} - \gamma_{(a)(b)(c)|(d)} \\
&+ \gamma_{(b)(a)(i)} \gamma_{(c)}{}^{(i)}{}_{(d)} - \gamma_{(b)(a)(i)} \gamma_{(d)}{}^{(i)}{}_{(c)} + \gamma_{(i)(a)(c)} \gamma_{(b)}{}^{(i)}{}_{(d)} - \gamma_{(i)(a)(d)} \gamma_{(b)}{}^{(i)}{}_{(c)}.
\end{aligned}
\tag{1.22}
$$

1.2.5 Bianchi identity in terms of Ricci Rotation coefficients

We write

$$
\begin{aligned}
R_{(a)(b)[(c)(d)|(f)]} =& R_{ij[kl;p]} e^i_{(a)} e^j_{(b)} e^k_{(c)} e^l_{(d)} e^p_{(f)} \\
=& \frac{1}{6} \sum_{[(c)(d)(f)]} \left(e^i_{(a)} e^j_{(b)} e^k_{(c)} e^l_{(d)} e^p_{(f)} \nabla_r R_{ijkl} \right) \\
=& \frac{1}{6} \sum_{[(c)(d)(f)]} \Big(R_{(a)(b)(c)(d),(f)} - \gamma^{(i)}{}_{(a)(f)} R_{(i)(b)(c)(d)} - \gamma^{(i)}{}_{(b)(f)} R_{(a)(i)(c)(d)} \\
&- \gamma^{(i)}{}_{(c)(f)} R_{(a)(b)(i)(d)} - \gamma^{(i)}{}_{(d)(f)} R_{(a)(b)(c)(i)} \Big).
\end{aligned}
\tag{1.23}
$$

The symbol $\sum_{[(c)(d)(f)]}$ means "sum over antisymmetric configurations of $(c)(d)(f)$". Another way of writing this is

$$
\sum_{[(c)(d)(f)]} = \sum_{(c)(d)(f)} \epsilon_{(c)(d)(f)}, \tag{1.24}
$$

but do *not* contract indices like in the Einstein summation convention.

1.2.6 Geodesic equation

The geodesic equation is

$$
u^i \nabla_i u_j = 0. \tag{1.25}
$$

From the definitions of the Ricci rotation coefficients we can write this as

$$
e^{(j)}_j u^{(i)} \left(\partial_{(i)} u_{(j)} - \eta^{(n)(m)} \gamma_{(m)(j)(i)} u_{(n)} \right) = 0. \tag{1.26}
$$

The geodesic equation for coordinates may be written as

$$
\frac{du_{(j)}}{d\tau} - \gamma_{(k)(j)(i)} u^{(i)} u^{(k)} = 0, \tag{1.27a}
$$

$$
\frac{dx^i}{d\tau} - e^i_{(i)} u^{(i)} = 0. \tag{1.27b}
$$

1.3 Newman-Penrose formalism

1.3.1 Basic setup

The Newman-Penrose formalism is a tetrad formalism with the following internal metric

$$\eta_{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1.28}$$

The four basis vectors have special names

$$l^i \equiv e^i_{(1)}, \qquad n^i \equiv e^i_{(2)}, \qquad m^i \equiv e^i_{(3)}, \qquad \overline{m}^i \equiv e^i_{(4)}. \tag{1.29}$$

Using $\eta_{(a)(b)}$, the we see that

$$e^{i(1)} = n^i, \qquad e^{i(2)} = l^i, \qquad e^{i(3)} = -\overline{m}^i, \qquad e^{i(4)} = -m^i. \tag{1.30}$$

The metric in terms of the NP vectors is

$$g_{ij} = n_i l_j + n_j l_i - m_i \overline{m}_j - \overline{m}_i m_j. \tag{1.31}$$

The vectors $\{n^i, l^i\}$ are null and real, and the vectors $\{m^i, \overline{m}^i\}$ are null, complex, and complex conjugates of each other. We can read off the normalization conditions from $\eta_{(a)(b)}$:

$$l_i m^i = l_i \overline{m}^i = n_i m^i = n_i \overline{m}^i = l_i l^i = n_i n^i = m_i m^i = \overline{m}_i \overline{m}^i = 0, \tag{1.32}$$

and

$$l_i n^i = -m_i \overline{m}^i = 1. \tag{1.33}$$

1.3.2 Definitions: directional derivatives and Ricci rotation coefficients

The directional derivatives have special names

$$D \equiv l^i \nabla_i, \qquad \Delta \equiv n^i \nabla_i, \qquad \delta \equiv m^i \nabla_i, \qquad \overline{\delta} \equiv \overline{m}^i \nabla_i, \tag{1.34}$$

as do the Ricci rotation coefficients

$$\begin{aligned}
&\kappa \equiv \gamma_{(3)(1)(1)}, && \sigma \equiv \gamma_{(3)(1)(3)}, \\
&\lambda \equiv \gamma_{(2)(4)(4)}, && \nu \equiv \gamma_{(2)(4)(2)}, \\
&\rho \equiv \gamma_{(3)(1)(4)}, && \mu \equiv \gamma_{(2)(4)(3)}, \\
&\tau \equiv \gamma_{(3)(1)(2)}, && \pi \equiv \gamma_{(2)(4)(1)}, \\
&\epsilon \equiv \frac{1}{2}\left(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}\right), && \gamma \equiv \frac{1}{2}\left(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}\right), \\
&\alpha \equiv \frac{1}{2}\left(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}\right), && \beta \equiv \frac{1}{2}\left(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}\right).
\end{aligned} \tag{1.35}$$

These definitions, combined with the antisymmetry of $\gamma_{(a)(b)(c)}$ in its first two indices and the $3 \leftrightarrow 4$ rule with complex conjugate (We can find the complex conjugates of these by setting $3 \to 4$ and $4 \to 3$, as those are the only complex quantities) , allows us to express all 24 Ricci rotation coefficients in terms of the above 12 complex Newman-Penrose scalars. For example we can invert the last four relationships to get

$$\begin{aligned} \gamma_{(2)(1)(1)} &= \epsilon + \overline{\epsilon}, & \gamma_{(3)(4)(1)} &= \epsilon - \overline{\epsilon}, \\ \gamma_{(2)(1)(2)} &= \gamma + \overline{\gamma}, & \gamma_{(3)(4)(2)} &= \gamma - \overline{\gamma}, \\ \gamma_{(2)(1)(4)} &= \alpha + \overline{\beta}, & \gamma_{(3)(4)(4)} &= \alpha - \overline{\beta}. \end{aligned} \tag{1.36}$$

1.3.3 Definitions: components of Weyl and Ricci tensors

We want to find the tetrad components of the curvature tensors. The Riemann tensor can be decomposed into the Weyl and Ricci tensors, and it is the components of those tensors that have special names. In four dimensions both tensors have 10 components, which are written as five complex scalars. First we list the Weyl scalars

$$\begin{aligned} \Psi_0 &\equiv -C_{pqrs} l^p m^q l^r m^s, \\ \Psi_1 &\equiv -C_{pqrs} l^p n^q l^r m^s, \\ \Psi_2 &\equiv -C_{pqrs} l^p m^q \overline{m}^r n^s, \\ \Psi_3 &\equiv -C_{pqrs} l^p n^q \overline{m}^r n^s, \\ \Psi_4 &\equiv -C_{pqrs} n^p \overline{m}^q n^r \overline{m}^s. \end{aligned} \tag{1.37}$$

Chandrasekhar [Cha02] describes a combinatorial procedure (which I implement in my Mathematica note) that allows one to write all the nonzero tetrad contractions of the Weyl tensor in terms of the above five complex Weyl scalars.

There are four real and three complex Ricci terms. **NOTE:** Newman and Penrose [NP62] use $R_{ij} = R^k{}_{ijk}$, while we use the conventions Chandrasekhar [Cha02]:$R_{ij} = R^k{}_{ikj}$. Thus, to make our NP equations look the same as in NP's original paper, we define the below terms with a global change of sign from what you will find in the original NP paper. Note also that there appears to be a global sign flip typo in Chandrasekhar's formulas that include the Ricci terms: he should define them as we do below to get the NP equations he later derives

$$\begin{aligned} \Phi_{00} &\equiv \frac{1}{2} R_{pq} l^p l^q, & \Phi_{22} &\equiv \frac{1}{2} R_{pq} n^p n^q, \\ \Phi_{02} &\equiv \frac{1}{2} R_{pq} m^p m^q, & \Phi_{20} &\equiv \frac{1}{2} R_{pq} \overline{m}^p \overline{m}^q, \\ \Phi_{11} &\equiv \frac{1}{4} R_{pq} \left(l^p n^q + m^p \overline{m}^q \right), & \Phi_{01} &\equiv \frac{1}{2} R_{pq} l^p m^q, \\ \Phi_{10} &\equiv \frac{1}{2} R_{pq} l^p \overline{m}^q, & \Phi_{12} &\equiv \frac{1}{2} R_{pq} n^p m^q, \\ \Phi_{21} &\equiv \frac{1}{2} R_{pq} n^p \overline{m}^q, & \Lambda &\equiv -\frac{1}{24} R. \end{aligned} \tag{1.38}$$

As the Ricci tensor is symmetric it is straightforward to write the rest of the Ricci tensor components in terms of the above NP scalars.

1.3.4 Derived relations: commutation relations

We obtain relations for commuting the directional derivatives through the relation

$$\left[e^i_{(a)}\nabla_i, e^j_{(b)}\nabla_j\right] = C^{(c)}{}_{(a)(b)}e^k_{(c)}\nabla_k = \left(\gamma_{(c)(b)(a)} - \gamma_{(c)(a)(b)}\right) e^{k(c)}\nabla_k. \tag{1.39}$$

An exercise in applying the definitions gives us

$$[D,\Delta] = -\left(\gamma+\overline{\gamma}\right) D - \left(\epsilon+\overline{\epsilon}\right)\Delta + \left(\pi+\overline{\tau}\right)\delta + \left(\overline{\pi}+\tau\right)\overline{\delta}, \tag{1.40a}$$

$$[D,\delta] = \left(\overline{\pi}-\beta-\overline{\alpha}\right) D - \kappa\Delta + \left(\epsilon-\overline{\epsilon}+\overline{\rho}\right)\delta + \sigma\overline{\delta}, \tag{1.40b}$$

$$[\Delta,\delta] = \overline{\nu}D + \left(-\tau+\overline{\alpha}+\beta\right)\Delta + \left(\gamma-\overline{\gamma}-\mu\right)\delta - \overline{\lambda\delta}, \tag{1.40c}$$

$$\left[\delta,\overline{\delta}\right] = \left(\mu-\overline{\mu}\right) D + \left(\rho-\overline{\rho}\right)\Delta + \left(-\alpha+\overline{\beta}\right)\delta + \left(\overline{\alpha}-\beta\right)\overline{\delta}. \tag{1.40d}$$

These are (1.303)-(1.306) in [Cha02].

1.3.5 Derived relations: components of Riemann tensor ("Ricci identities")

Using Eq. (1.22) for the tetrad projections of the Riemann tensor, and the definitions (1.35) for the Ricci rotation coefficients, we can write down the independent components of the Riemann tensor. We can equate these expression to the components of the Riemann tensor written in terms of the NP scalars Ψ_0, Φ_{00}, etc, using Eq. (1.7).

We implement this in Mathematica and do not include the output here. See also [Cha02], Chptr 1, Eq.(310), and [NP62] (note though that NP use opposite signs for Ricci rotation coefficients than Chandrasekhar). These essentially give us transport equations for the Ricci rotation coefficients α, etc.

By taking linear combinations of these equations (and complex conjugates of them) we can eliminate some of the scalars Φ_{00}, etc. and get the so-call "eliminant relations" ([Cha02], Chptr 1, Eq.(310)). The Ricci identities as derived in the Mathematica notebook are (this numbering follows [Cha02], Eq. 310, and the equations listed here should exactly match those)

$$\kappa\left(-3\alpha-\overline{\beta}+\pi\right) + \rho\left(\epsilon+\overline{\epsilon}+\rho\right) + \sigma\overline{\sigma} - \overline{\kappa}\tau + \Phi_{00} - D\left(\rho\right) + \overline{\delta}\left(\kappa\right) = 0, \tag{1.41a}$$

$$\left(3\epsilon-\overline{\epsilon}+\rho+\overline{\rho}\right)\sigma - \kappa\left(\overline{\alpha}+3\beta-\overline{\pi}+\tau\right) + \Psi_0 - D\left(\sigma\right) + \delta\left(\kappa\right) = 0, \tag{1.41b}$$

$$-\left(\left(3\gamma+\overline{\gamma}\right)\kappa\right) + \overline{\pi}\rho + \left(\epsilon-\overline{\epsilon}+\rho\right)\tau + \sigma\left(\pi+\overline{\tau}\right) + \Phi_{01} + \Psi_1 - D\left(\tau\right) + \Delta\left(\kappa\right) = 0, \tag{1.41c}$$

$$-\left(\overline{\beta}\epsilon\right) - \gamma\overline{\kappa} - \kappa\lambda + \pi\left(\epsilon+\rho\right) + \alpha\left(-2\epsilon+\overline{\epsilon}+\rho\right) + \beta\overline{\sigma} + \Phi_{10} - D\left(\alpha\right) + \overline{\delta}\left(\epsilon\right) = 0, \tag{1.41d}$$

$$\kappa\left(\gamma+\mu\right) + \epsilon\left(\overline{\alpha}-\overline{\pi}\right) + \beta\left(\overline{\epsilon}-\overline{\rho}\right) - \left(\alpha+\pi\right)\sigma - \Psi_1 + D\left(\beta\right) - \delta\left(\epsilon\right) = 0, \tag{1.41e}$$

$$2\gamma\epsilon+\overline{\gamma}\epsilon+\gamma\overline{\epsilon}+\Lambda+\kappa\nu-\pi\left(\beta+\tau\right)-\alpha\left(\overline{\pi}+\tau\right)-\beta\overline{\tau}-\Phi_{11}-\Psi_2+D\left(\gamma\right)-\Delta\left(\epsilon\right)=0, \tag{1.41f}$$

$$3\epsilon\lambda+\overline{\kappa}\nu-\pi\left(\alpha-\overline{\beta}+\pi\right)-\lambda\left(\overline{\epsilon}+\rho\right)-\mu\overline{\sigma}-\Phi_{20}+D\left(\lambda\right)-\overline{\delta}\left(\pi\right)=0, \tag{1.41g}$$

$$-2\Lambda+\kappa\nu+\overline{\alpha}\pi-\pi\left(\beta+\overline{\pi}\right)+\mu\left(\epsilon+\overline{\epsilon}-\overline{\rho}\right)-\lambda\sigma-\Psi_2+D\left(\mu\right)-\delta\left(\pi\right)=0, \tag{1.41h}$$

$$\left(3\epsilon+\overline{\epsilon}\right)\nu-\gamma\pi+\overline{\gamma}\pi-\lambda\left(\overline{\pi}+\tau\right)-\mu\left(\pi+\overline{\tau}\right)-\Phi_{21}-\Psi_3+D\left(\nu\right)-\Delta\left(\pi\right)=0, \tag{1.41i}$$

$$\lambda\left(3\gamma-\overline{\gamma}+\mu+\overline{\mu}\right)-\nu\left(3\alpha+\overline{\beta}+\pi-\overline{\tau}\right)+\Psi_4+\Delta\left(\lambda\right)-\overline{\delta}\left(\nu\right)=0, \tag{1.41j}$$

$$\kappa\left(-\mu+\overline{\mu}\right)-\left(\overline{\alpha}+\beta\right)\rho+3\alpha\sigma-\overline{\beta}\sigma+\left(-\rho+\overline{\rho}\right)\tau-\Phi_{01}+\Psi_1+\delta\left(\rho\right)-\overline{\delta}\left(\sigma\right)=0, \tag{1.41k}$$

$$-\left(\alpha\overline{\alpha}\right)+2\alpha\beta-\beta\overline{\beta}-\Lambda+\epsilon\left(-\mu+\overline{\mu}\right)-\left(\gamma+\mu\right)\rho+\gamma\overline{\rho}+\lambda\sigma-\Phi_{11}+\Psi_2+\delta\left(\alpha\right)-\overline{\delta}\left(\beta\right)=0, \tag{1.41l}$$

$$-\left(\overline{\alpha}\lambda\right)+3\beta\lambda-\left(\alpha+\overline{\beta}\right)\mu+\left(-\mu+\overline{\mu}\right)\pi-\nu\rho+\nu\overline{\rho}-\Phi_{21}+\Psi_3+\delta\left(\lambda\right)-\overline{\delta}\left(\mu\right)=0, \tag{1.41m}$$

$$-\left(\lambda\overline{\lambda}\right)-\mu\left(\gamma+\overline{\gamma}+\mu\right)+\overline{\nu}\pi+\nu\left(\overline{\alpha}+3\beta-\tau\right)-\Phi_{22}+\delta\left(\nu\right)-\Delta\left(\mu\right)=0, \tag{1.41n}$$

$$\overline{\alpha}\gamma+2\beta\gamma-\beta\overline{\gamma}-\alpha\overline{\lambda}-\beta\mu+\epsilon\overline{\nu}+\nu\sigma-\left(\gamma+\mu\right)\tau-\Phi_{12}+\delta\left(\gamma\right)-\Delta\left(\beta\right)=0, \tag{1.41o}$$

$$\kappa\overline{\nu}-\overline{\lambda}\rho+3\gamma\sigma-\left(\overline{\gamma}+\mu\right)\sigma+\overline{\alpha}\tau-\tau\left(\beta+\tau\right)-\Phi_{02}+\delta\left(\tau\right)-\Delta\left(\sigma\right)=0, \tag{1.41p}$$

$$2\Lambda-\kappa\nu-\left(\gamma+\overline{\gamma}-\overline{\mu}\right)\rho+\lambda\sigma+\tau\left(\alpha-\overline{\beta}+\overline{\tau}\right)+\Psi_2+\Delta\left(\rho\right)-\overline{\delta}\left(\tau\right)=0, \tag{1.41q}$$

$$\alpha\left(\overline{\gamma}-\overline{\mu}\right)+\nu\left(\epsilon+\rho\right)-\lambda\left(\beta+\tau\right)+\gamma\left(\overline{\beta}-\overline{\tau}\right)-\Psi_3-\Delta\left(\alpha\right)+\overline{\delta}\left(\gamma\right)=0. \tag{1.41r}$$

1.3.6 Derived: Weyl scalars in terms of the Ricci rotation coefficients

We can think of the results in Sec. 1.3.5 as providing definitions for the Weyl scalars in terms of derivatives and polynomial combinations of the Ricci rotation coefficients. We have

$$\Psi_0=\overline{\alpha}\kappa+3\beta\kappa-\kappa\overline{\pi}-3\epsilon\sigma+\overline{\epsilon}\sigma-\rho\sigma-\overline{\rho}\sigma+\kappa\tau+D\left(\sigma\right)-\delta\left(\kappa\right), \tag{1.42a}$$

$$\Psi_1=\overline{\alpha}\epsilon+\beta\overline{\epsilon}+\gamma\kappa+\kappa\mu-\epsilon\overline{\pi}-\beta\overline{\rho}-\alpha\sigma-\pi\sigma+D\left(\beta\right)-\delta\left(\epsilon\right), \tag{1.42b}$$

$$\Psi_2=-2\Lambda+\epsilon\mu+\overline{\epsilon}\mu+\kappa\nu+\overline{\alpha}\pi-\beta\pi-\pi\overline{\pi}-\mu\overline{\rho}-\lambda\sigma+D\left(\mu\right)-\delta\left(\pi\right), \tag{1.42c}$$

$$\Psi_3=\overline{\beta}\gamma+\alpha\overline{\gamma}-\beta\lambda-\alpha\overline{\mu}+\epsilon\nu+\nu\rho-\lambda\tau-\gamma\overline{\tau}-\Delta\left(\alpha\right)+\overline{\delta}\left(\gamma\right), \tag{1.42d}$$

$$\Psi_4=-3\gamma\lambda+\overline{\gamma}\lambda-\lambda\mu-\lambda\overline{\mu}+3\alpha\nu+\overline{\beta}\nu+\nu\pi-\nu\overline{\tau}-\Delta\left(\lambda\right)+\overline{\delta}\left(\nu\right). \tag{1.42e}$$

These equations are highlighted in the Mathematica note in the computation of the Ricci identities.

1.3.7 Derived: Components of the Bianchi identities

Using Eq. (1.23) and the definitions for the NP scalars for the Ricci rotation coefficients, we can derive long expressions for the Bianchi identities. We can obtain shorter expressions by writing the Riemann tensor in terms of the Weyl and Ricci scalars using Eq. (1.7). The Bianchi identities then become first order differential equations for those scalars quantities. These are again derived in the Mathematica note. The Bianchi identities written in this form are (c.f. Eq. (321) in [Cha02])

$$4\alpha\Psi_0 - \pi\Psi_0 - 2\left(\epsilon + 2\rho\right)\Psi_1 + 3\kappa\Psi_2 + D\left(\Psi_1\right) - \overline{\delta}\left(\Psi_0\right) = \mathcal{R}_a, \tag{1.43a}$$

$$-\left(\lambda\Psi_0\right) - 2\alpha\Psi_1 + 2\pi\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 - D\left(\Psi_2\right) + \overline{\delta}\left(\Psi_1\right) = \mathcal{R}_b, \tag{1.43b}$$

$$2\lambda\Psi_1 - 3\pi\Psi_2 + 2\epsilon\Psi_3 - 2\rho\Psi_3 + \kappa\Psi_4 + D\left(\Psi_3\right) - \overline{\delta}\left(\Psi_2\right) = \mathcal{R}_c, \tag{1.43c}$$

$$-3\lambda\Psi_2 + 2\alpha\Psi_3 + 4\pi\Psi_3 - 4\epsilon\Psi_4 + \rho\Psi_4 - D\left(\Psi_4\right) + \overline{\delta}\left(\Psi_3\right) = \mathcal{R}_d, \tag{1.43d}$$

$$4\gamma\Psi_0 - \mu\Psi_0 - 2\left(\beta + 2\tau\right)\Psi_1 + 3\sigma\Psi_2 + \delta\left(\Psi_1\right) - \Delta\left(\Psi_0\right) = \mathcal{R}_e, \tag{1.43e}$$

$$\nu\Psi_0 + 2\gamma\Psi_1 - 2\mu\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + \delta\left(\Psi_2\right) - \Delta\left(\Psi_1\right) = \mathcal{R}_f, \tag{1.43f}$$

$$2\nu\Psi_1 - 3\mu\Psi_2 + 2\beta\Psi_3 - 2\tau\Psi_3 + \sigma\Psi_4 + \delta\left(\Psi_3\right) - \Delta\left(\Psi_2\right) = \mathcal{R}_g, \tag{1.43g}$$

$$3\nu\Psi_2 - 2\gamma\Psi_3 - 4\mu\Psi_3 + 4\beta\Psi_4 - \tau\Psi_4 + \delta\left(\Psi_4\right) - \Delta\left(\Psi_3\right) = \mathcal{R}_h, \tag{1.43h}$$

where the $\mathcal{R}$ are Ricci scalar terms

$$\begin{aligned}\mathcal{R}_a \equiv & \left(2\left(\overline{\alpha} + \beta\right) - \overline{\pi}\right)\Phi_{00} - 2\left(\epsilon + \overline{\rho}\right)\Phi_{01} + \overline{\kappa}\Phi_{02} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} \\ & + D\left(\Phi_{01}\right) - \delta\left(\Phi_{00}\right),\end{aligned} \tag{1.44a}$$

$$\begin{aligned}\mathcal{R}_b \equiv & \left(-2\left(\gamma + \overline{\gamma}\right) + \overline{\mu}\right)\Phi_{00} + 2\left(\alpha + \overline{\tau}\right)\Phi_{01} - \overline{\sigma}\Phi_{02} + 2\tau\Phi_{10} - 2\rho\Phi_{11} \\ & + 2D\left(\Lambda\right) + \Delta\left(\Phi_{00}\right) - \overline{\delta}\left(\Phi_{01}\right),\end{aligned} \tag{1.44b}$$

$$\begin{aligned}\mathcal{R}_c \equiv & 2\mu\Phi_{10} - 2\pi\Phi_{11} + 2\overline{\alpha}\Phi_{20} - \left(2\beta + \overline{\pi}\right)\Phi_{20} + 2\epsilon\Phi_{21} - 2\overline{\rho}\Phi_{21} + \overline{\kappa}\Phi_{22} \\ & + D\left(\Phi_{21}\right) - \delta\left(\Phi_{20}\right) + 2\overline{\delta}\left(\Lambda\right),\end{aligned} \tag{1.44c}$$

$$\begin{aligned}\mathcal{R}_d \equiv & -2\nu\Phi_{10} + 2\lambda\Phi_{11} + \left(2\gamma - 2\overline{\gamma} + \overline{\mu}\right)\Phi_{20} - 2\alpha\Phi_{21} + 2\overline{\tau}\Phi_{21} - \overline{\sigma}\Phi_{22} \\ & + \Delta\left(\Phi_{20}\right) - \overline{\delta}\left(\Phi_{21}\right),\end{aligned} \tag{1.44d}$$

$$\begin{aligned}\mathcal{R}_e \equiv & \overline{\lambda}\Phi_{00} + 2\left(\beta - \overline{\pi}\right)\Phi_{01} - \left(2\epsilon - 2\overline{\epsilon} + \overline{\rho}\right)\Phi_{02} - 2\sigma\Phi_{11} + 2\kappa\Phi_{12} \\ & + D\left(\Phi_{02}\right) - \delta\left(\Phi_{01}\right),\end{aligned} \tag{1.44e}$$

$$\begin{aligned}\mathcal{R}_f \equiv & \overline{\nu}\Phi_{00} + 2\left(\gamma - \overline{\mu}\right)\Phi_{01} - \left(2\alpha - 2\overline{\beta} + \overline{\tau}\right)\Phi_{02} - 2\tau\Phi_{11} + 2\rho\Phi_{12} \\ & - 2\delta\left(\Lambda\right) - \Delta\left(\Phi_{01}\right) + \overline{\delta}\left(\Phi_{02}\right),\end{aligned} \tag{1.44f}$$

$$\begin{aligned}\mathcal{R}_g \equiv & 2\mu\Phi_{11} - 2\pi\Phi_{12} + \overline{\lambda}\Phi_{20} - 2\left(\beta + \overline{\pi}\right)\Phi_{21} + \left(2\left(\epsilon + \overline{\epsilon}\right) - \overline{\rho}\right)\Phi_{22} \\ & + D\left(\Phi_{22}\right) - \delta\left(\Phi_{21}\right) + 2\Delta\left(\Lambda\right),\end{aligned} \tag{1.44g}$$

$$\mathcal{R}_h \equiv 2\nu\Phi_{11} - 2\lambda\Phi_{12} + \overline{\nu}\Phi_{20} - 2\left(\gamma + \overline{\mu}\right)\Phi_{21} + \left(2\left(\alpha + \overline{\beta}\right) - \overline{\tau}\right)\Phi_{22}$$

$$- \Delta(\Phi_{21}) + \bar{\delta}(\Phi_{22}). \tag{1.44h}$$

The Ricci scalar terms are all zero for vacuum spacetime solutions to the Einstein equations.

1.3.8 Transformations of null frame

There are $4\times 4 = 16$ degrees of freedom with the choice of the four null vectors $\{l^i, n^i, m^i, \overline{m}^i\}$. The 10 conditions

$$\begin{aligned} l_i l^i = n_i n^i = m_i m^i = \overline{m}_i \overline{m}^i = l_i m^i = l_i \overline{m}^i = n_i m^i = n_i \overline{m}^i = 0, \\ l_i n^i = -m_i \overline{m}^i = 1, \end{aligned} \tag{1.45}$$

leave us with 6 transformations, which we can identify as the Lorentz transformations ("rotations") at any given tangent space. Note that picking a null frame is independent of picking a set of coordinates/gauge for the manifold. When working in the NP formalism we have 6 choices of null frame transformations and 4 choices of gauge transformations.

Following [Cha02], we consider three classes of transformations that preserve the above constraints

I rotations which leave l unchanged

$$\begin{aligned} l^i &\to l^i, \\ n^i &\to n^i + \bar{a} m^i + a \overline{m}^i + a\bar{a} l^i, \\ m^i &\to m^i + a l^i. \end{aligned} \tag{1.46}$$

II rotations which leave n unchanged

$$\begin{aligned} n^i &\to n^i, \\ l^i &\to l^i + \bar{b} m^i + b \overline{m}^i + b\bar{b} n^i, \\ m^i &\to m^i + b n^i. \end{aligned} \tag{1.47}$$

III rotations which leave directions of l and n unchanged and rotate the vectors m and $\overline{m}$

$$\begin{aligned} l^i &\to A^{-1} n^i, \\ n^i &\to A n^i, \\ m^i &\to e^{i\theta} m^i. \end{aligned} \tag{1.48}$$

These three classes form a basis for the 6 Lorentz transformations: two complex numbers a and b, and two real numbers A and θ. We can determine how the Weyl scalars, etc. transform under the above transformations using their definitions. We list how the Weyl scalars transform as this will be important for the Petrov classification of spacetimes. In deriving these relations we used the fact that, due to the symmetries of the Weyl tensor, many components are zero (this can be found in the Mathematica note).

Class I:

$$\begin{aligned}
&\Psi_0 \to \Psi_0,\\
&\Psi_1 \to \Psi_1 + \overline{a}\Psi_0,\\
&\Psi_2 \to \Psi_2 + 2\overline{a}\Psi_1 + (\overline{a})^2\Psi_0,\\
&\Psi_3 \to \Psi_3 + 2\overline{a}\Psi_2 + 3(\overline{a})^2\Psi_1 + (\overline{a})^3\Psi_0,\\
&\Psi_4 \to \Psi_4 + 4\overline{a}\Psi_3 + 6(\overline{a})^2\Psi_2 + 4(\overline{a})^3\Psi_1 + (\overline{a})^4\Psi_0.
\end{aligned} \tag{1.49}$$

Class II:

$$\begin{aligned}
&\Psi_0 \to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4,\\
&\Psi_1 \to \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4,\\
&\Psi_2 \to \Psi_2 + 2b\Psi_3 + b^2\Psi_4,\\
&\Psi_3 \to \Psi_3 + b\Psi_4,\\
&\Psi_4 \to \Psi_4.
\end{aligned} \tag{1.50}$$

Class III:

$$\begin{aligned}
&\Psi_0 \to A^{-2}e^{2i\theta}\Psi_0,\\
&\Psi_1 \to A^{-1}e^{i\theta}\Psi_1,\\
&\Psi_2 \to \Psi_2,\\
&\Psi_3 \to Ae^{-i\theta}\Psi_3,\\
&\Psi_4 \to A^2e^{-2i\theta}\Psi_4.
\end{aligned} \tag{1.51}$$

1.3.9 Invariance under swapping of null frame

The NP equations taken as a set are invariant under [GHP73]

$$l^i \leftrightarrow n^i \qquad m^i \leftrightarrow \overline{m}^i. \tag{1.52}$$

Under this transformation we have

$$D \leftrightarrow \Delta, \qquad\qquad \delta \leftrightarrow \overline{\delta}, \tag{1.53}$$

and

$$\begin{aligned}
&\kappa \leftrightarrow -\nu, \quad \sigma \leftrightarrow -\lambda,\\
&\rho \leftrightarrow -\mu, \quad \tau \leftrightarrow -\pi,\\
&\epsilon \leftrightarrow -\gamma, \quad \alpha \leftrightarrow -\beta,
\end{aligned} \tag{1.54}$$

and

$$\Psi_0 \leftrightarrow \Psi_4, \qquad \Psi_1 \leftrightarrow \Psi_3, \qquad \Psi_2 \leftrightarrow \Psi_2. \tag{1.55}$$

1.4 Spin and boost weight of the NP scalars

GHP [GHP73] first pointed out that we can classify the NP scalars (or algebraic combinations of them) into quantities of definite spin (and 'boost') weight. As this is important for how we expand the NP scalar in terms of spin weighted spherical harmonics, we go over some of the main parts of their formalism here.

We classify the spin and boost weight of a NP scalar by considering how it transforms under the following complex rotation

$$l^i \to \xi\overline{\xi} l^i, \tag{1.56a}$$

$$n^i \to \xi^{-1}\overline{\xi}^{-1} l^i, \tag{1.56b}$$

$$m^i \to \xi\overline{\xi}^{-1} m^i, \tag{1.56c}$$

$$\overline{m}^i \to \xi^{-1}\overline{\xi}\overline{m}^i, \tag{1.56d}$$

where ξ is a complex number. A scalar that transforms as

$$\eta \to \lambda^p \overline{\lambda}^q \eta, \tag{1.57}$$

has weight $\{p, q\}$, spin weight $(p-q)/2$ and boost weight $(p+q)/2$. From this definition, we see that the NP scalar of definite weight are

NP scalar	weight	spin weight	boost weight
Ψ_0	$\{4, 0\}$	2	2
Ψ_1	$\{2, 0\}$	1	1
Ψ_2	$\{0, 0\}$	0	0
Ψ_3	$\{-2, 0\}$	-1	-1
Ψ_4	$\{-4, 0\}$	-2	-2
σ	$\{3, -1\}$	2	1
κ	$\{3, 1\}$	1	2
τ	$\{1, -1\}$	1	0
ρ	$\{1, 1\}$	0	1
μ	$\{-1, -1\}$	0	-1
π	$\{-1, 1\}$	-1	0
ν	$\{-3, -1\}$	-1	-2
λ	$\{-3, 1\}$	-2	-1

The NP scalars of indefinite weight are $\{\alpha, \beta, \gamma, \epsilon\}$, as $\nabla_i \xi \neq 0$, so if we have e.g. $n^i \nabla_k l_i$, then the Ricci rotation coefficient would have terms like $\nabla_k \left(\xi\overline{\xi}\right)$. Note as well that the derivative operators $\{D, \Delta, \delta, \overline{\delta}\}$ have definite spin weight when considered by themselves, but we cannot think of them as having definite spin weight when they act of a scalar. To make them operators of definite spin weight, acting on a scalar of weight $\{p, q\}$ we define

$$\text{þ}\eta \equiv \left(D - p\epsilon - q\overline{\epsilon}\right)\eta, \tag{1.58a}$$

$$\text{þ}'\eta \equiv (\Delta - p\gamma - q\overline{\gamma})\,\eta, \tag{1.58b}$$
$$\eth\eta \equiv (\delta - p\beta - q\overline{\alpha})\,\eta, \tag{1.58c}$$
$$\eth'\eta \equiv \left(\overline{\delta} - p\alpha - q\overline{\beta}\right)\eta. \tag{1.58d}$$

These operators have the definite spin and boost weight

operator	weight	spin weight	boost weight
þ	$\{1,1\}$	0	1
þ$'$	$\{-1,-1\}$	0	-1
$\eth$	$\{1,-1\}$	1	0
$\eth'$	$\{-1,1\}$	-1	0

Finally, we note that there is a physical interpretation for spin weight: spin 0 captures “scalar” information, spin 1 captures “vector” information, and spin 2 captures “tensor” information.

1.5 Interpretation of the NP scalars

We can interpret several of the NP scalars using the Raychaudhuri equations of null congruences of the n^i, l^i null vectors. These interpretations prove to be extremely useful when using NP equations.

1.5.1 Derivatives of NP vectors

First we write out

$$\nabla_i e_{(a)j} = e_i^{(b)} e_{(b)}^k \nabla_j e_{(a)j} = \gamma_{(c)(a)(b)} e_i^{(b)} e_j^{(c)}. \tag{1.59}$$

From this and the NP relations, we have

$$\begin{aligned}
\nabla_i l_j =& -\left(l_j \overline{m}_i \left(\overline{\alpha} + \beta\right)\right) - l_j m_i \left(\alpha + \overline{\beta}\right) + l_i l_j \left(\gamma + \overline{\gamma}\right) + l_j n_i \left(\epsilon + \overline{\epsilon}\right) - \overline{m}_j n_i \kappa \\
& - m_j n_i \overline{\kappa} + \overline{m}_j m_i \rho + \overline{m}_i m_j \overline{\rho} + \overline{m}_i \overline{m}_j \sigma + m_i m_j \overline{\sigma} - l_i \overline{m}_j \tau - l_i m_j \overline{\tau}, \qquad (1.60a) \\
\nabla_i n_j =& m_i n_j \alpha + \overline{m}_i n_j \overline{\alpha} + \overline{m}_i n_j \beta + m_i n_j \overline{\beta} - l_i n_j \gamma - l_i n_j \overline{\gamma} - n_i n_j \epsilon - n_i n_j \overline{\epsilon} - m_i m_j \lambda \\
& - \overline{m}_i \overline{m}_j \overline{\lambda} - \overline{m}_i m_j \mu - \overline{m}_j m_i \overline{\mu} + l_i m_j \nu + l_i \overline{m}_j \overline{\nu} + m_j n_i \pi + \overline{m}_j n_i \overline{\pi}, \qquad (1.60b) \\
\nabla_i m_j =& -\left(m_i m_j \alpha\right) + \overline{m}_i m_j \overline{\alpha} - \overline{m}_i m_j \beta + m_i m_j \overline{\beta} + l_i m_j \gamma - l_i m_j \overline{\gamma} + m_j n_i \epsilon - m_j n_i \overline{\epsilon} \\
& - n_i n_j \kappa - l_j \overline{m}_i \overline{\lambda} - l_j m_i \overline{\mu} + l_i l_j \overline{\nu} + l_j n_i \overline{\pi} + m_i n_j \rho + \overline{m}_i n_j \sigma - l_i n_j \tau. \qquad (1.60c)
\end{aligned}$$

1.5.2 Geodesics

From the equations for l and n we have

$$l^i \nabla_i l_j = \left(\epsilon + \overline{\epsilon}\right) l_j - \kappa \overline{m}_j - \overline{\kappa} m_j$$

$$n^i \nabla_i n_j = -\left(\gamma + \overline{\gamma}\right) n_j + \nu m_j + \overline{\nu m}_j. \tag{1.61}$$

We see for l^i/n^i to be pregeodesic we need $\kappa = 0/\nu = 0$. If we want l^i/n^i to be geodesic we additionally need $\Re\epsilon = 0/\Re\gamma = 0$.

1.6 Petrov classification

From the Eqs. (1.49), (1.50), and (1.51), we see that at any given tangent space it may be possible to set some of the Weyl scalars to zero through a choice of null frame transformation. From the perspective of the NP formalism, we can think of the Petrov classification of spacetimes as enumerating how many Weyl scalars can be sent to zero through a null frame transformation.

Chapter 2

Coordinates and null tetrads for Kerr

These are collected equations on various tetrads and coordinates for the Kerr spacetimes. We begin with the Kinnersley tetrad, and by the end of the chapter we have a tetrad that is regular both at the black hole horizon and future null infinity. See also [RLGP21]. The black hole mass is M, and the black hole spin is a. **NOTE:** we use the non-standard symbol "varpi" $\varpi = 3.14...$, and reserve use of π for the Newman-Penrose scalar.

2.1 Kerr in Boyer-Lindquist coordinates

2.1.1 Setup

The Kerr spacetime in Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 = &\left(1 - \frac{2Mr}{\Sigma_{BL}}\right) dt^2 + 2\left(\frac{2Mar\sin^2\vartheta}{\Sigma_{BL}}\right) dt d\varphi - \frac{\Sigma_{BL}}{\Delta_{BL}} dr^2 \\ &- \Sigma_{BL} d\vartheta^2 - \sin^2\vartheta \left(r^2 + a^2 + 2Ma^2 r \frac{\sin^2\vartheta}{\Sigma_{BL}}\right) d\varphi^2, \end{aligned} \tag{2.1}$$

where

$$\Sigma_{BL} \equiv r^2 + a^2\cos^2\vartheta, \tag{2.2a}$$

$$\Delta_{BL} \equiv r^2 - 2Mr + a^2. \tag{2.2b}$$

2.1.2 The tetrad

The Kinnersley tetrad in Boyer-Lindquist coordinates is

$$l^i_{Kin} = \left(\frac{r^2 + a^2}{\Delta_{BL}}, 1, 0, \frac{a}{\Delta_{BL}}\right), \tag{2.3a}$$

$$n^i_{Kin} = \frac{1}{2\Sigma_{BL}} \left(r^2 + a^2, -\Delta_{BL}, 0, a\right), \tag{2.3b}$$

$$m^i_{Kin} = i\frac{1}{2^{1/2}\left(r+ia\cos\vartheta\right)}\left(ia\sin\vartheta, 0, 1, \frac{i}{\sin\vartheta}\right). \tag{2.3c}$$

The event horizon is at $(\nabla r)^2 = 0$; that is at the point $\Delta_{BL} = 0$. This function has two zeros, which define the outer and inner horizons

$$r_{ou} = M \pm \left(M^2 - a^2\right)^{1/2}. \tag{2.4a}$$

2.2 Kerr in ingoing Eddington-Finkelstein coordinates

2.2.1 Setup

We transform

$$dv \equiv dt + dr_* - dr, \tag{2.5a}$$
$$d\phi \equiv d\varphi + \frac{a}{r^2+a^2}dr_*. \tag{2.5b}$$

where

$$\frac{dr_*}{dr} \equiv \frac{r^2+a^2}{\Delta_{BL}}. \tag{2.6}$$

The Kerr metric in ingoing Eddington-Finkelstein coordinates is

$$\begin{aligned} ds^2 =& \left(1-\frac{2Mr}{\Sigma_{BL}}\right)dv^2 - \frac{4Mr}{\Sigma_{BL}}\left(dr - a\sin^2\vartheta d\phi\right)dv \\ &- \left(1+\frac{2Mr}{\Sigma_{BL}}\right)\left(dr^2 - 2a\sin^2\vartheta dr d\phi\right) \\ &- \Sigma d\vartheta^2 - \left(a^2 + r^2 + 2Mr\frac{a^2}{\Sigma_{BL}}\sin^2\vartheta\right)d\phi^2. \end{aligned} \tag{2.7}$$

2.2.2 The tetrad

The Kinnersely tetrad transforms as

$$v^i \to \frac{\partial x^i}{\partial y^j}v^j, \tag{2.8}$$

so that

$$l^i_{Kin} = \left(1+\frac{4Mr}{\Delta_{BL}}, 1, 0, \frac{2a}{\Delta_{BL}}\right), \tag{2.9a}$$
$$n^i_{Kin} = \frac{1}{2\Sigma_{BL}}\left(\Delta_{BL}, -\Delta_{BL}, 0, 0\right), \tag{2.9b}$$

$$m^i_{Kin} = \frac{1}{2^{1/2}\left(r + ia\cos\vartheta\right)}\left(ia\sin\vartheta, 0, 1, \frac{i}{\sin\vartheta}\right). \tag{2.9c}$$

This tetrad is singular at the black hole horizons. This coordinate singularity can be removed by the following tetrad transformation [Teu73]

$$l^i \to \Delta_{BL} l^i, \tag{2.10a}$$
$$n^i \to \frac{1}{\Delta_{BL}} n^i, \tag{2.10b}$$

but this spoils the property that $\epsilon = 0$ (which holds in the Kinnersely tetrad). We rotate the tetrad to set $\gamma = 0$, which is more useful for metric reconstruction in outgoing radiation gauge

$$l^i \to \frac{\Delta_{BL}}{2\Sigma_{BL}} l^i, \tag{2.11a}$$
$$n^i \to \frac{2\Sigma_{BL}}{\Delta_{BL}} n^i, \tag{2.11b}$$
$$m^i \to \exp\left[-2i\arctan\left[\frac{r}{a\sin\vartheta}\right]\right] m^i, \tag{2.11c}$$

we obtain

$$l^i = \left(\frac{r^2 + 2Mr + a^2}{2\Sigma_{BL}}, \frac{\Delta_{BL}}{2\Sigma_{BL}}, 0, \frac{a}{2\Sigma_{BL}}\right), \tag{2.12a}$$
$$n^i = \left(1, -1, 0, 0\right), \tag{2.12b}$$
$$m^i = \frac{1}{2^{1/2}\left(r - ia\cos\vartheta\right)}\left(-ia\sin\vartheta, 0, -1, -\frac{i}{\sin\vartheta}\right). \tag{2.12c}$$

2.3 Kerr in ingoing Eddington-Finkelstein coordinates with hyperboloidal compactification

2.3.1 Setup

For further discussion of hyperboloidal compactification see, e.g. [Zen08]. The ingoing/outgoing radial null characteristic speeds for Kerr in ingoing Eddington-Finkelstein coordinates are

$$c_+ = 1 - \frac{4Mr}{2Mr + \Sigma_{BL}}, \tag{2.13a}$$
$$c_- = -1. \tag{2.13b}$$

We do not need to consider angular characteristic speeds as those die off more quickly as we go to future null infinity/spatial infinity. We choose a radial compactification and time rescaling $R(r)$ and $T(v,r)$, respectively. The ingoing/outgoing radial null characteristic speeds are now

$$\tilde{c}_{\pm} = \frac{dR/dr}{\frac{1}{c_{\pm}}\partial_v T + \partial_r T}. \tag{2.14}$$

We want to choose a time function that sets $\tilde{c}_-|_{r=\infty} = 0$ while keeping $0 < \tilde{c}_+|_{r=\infty} < \infty$. We choose the time coordinate to be of the form

$$T(v,r) = v + h(r), \tag{2.15}$$

and a compactification [1]

$$R(r) \equiv \frac{L^2}{r}, \tag{2.16}$$

where L is a constant. Series expanding about $r = \infty$, we have

$$\tilde{c}_+ = \left(1 + \frac{4M}{r} + \frac{8M^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) + \frac{dh}{dr}\right)^{-1} \left(-\frac{L^2}{r^2}\right), \tag{2.17a}$$

$$\tilde{c}_- = \left(-1 + \frac{dh}{dr}\right)^{-1} \left(-\frac{L^2}{r^2}\right). \tag{2.17b}$$

We see that the choice

$$\frac{dh}{dr} = -1 - \frac{4M}{r}, \tag{2.18}$$

sets $\tilde{c}_-|_{R=0} = 0$ while keeping $0 > \tilde{c}_+|_{R=0} > -\infty$ (our choice of compactification flips the signs of the ingoing and outgoing characteristics, and $r = \infty$ is mapped to $R = 0$). We can say that $R = 0$ is located "on" future null infinity of Kerr.

2.3.2 The tetrad

We apply the above transformations $T(v,r)$, $R(r)$, to Eq. (2.12) to obtain

$$l^i = \frac{R^2}{L^4 + a^2R^2\cos^2\vartheta}\left(2M\left(2M - \left(\frac{a}{L}\right)^2 R\right), -\frac{1}{2}\left(L^2 - 2MR + \left(\frac{a}{L}\right)^2 R^2\right), 0, a\right), \tag{2.19a}$$

$$n^i = \left(2 + \frac{4MR}{L^2}, \frac{R^2}{L^2}, 0, 0\right), \tag{2.19b}$$

$$m^i = \frac{R}{2^{1/2}\left(L^2 - iaR\cos\vartheta\right)}\left(-ia\sin\vartheta, 0, -1, -\frac{i}{\sin\vartheta}\right). \tag{2.19c}$$

[1] We choose this compactification as it is straightforward to manipulate.

2.3.3 Weyl scalars and Ricci rotation coefficients

The nonzero Weyl scalar is

$$\Psi_2 = -\frac{MR^3}{\left(L^2 - iaR\cos\left(\vartheta\right)\right)^3}. \tag{2.20}$$

and the nonzero Ricci rotation coefficients are .

$$\rho = -\frac{R\left(a^2R^2 + L^4 - 2L^2MR\right)}{2\left(L^2 - iaR\cos(\vartheta)\right)^2\left(L^2 + iaR\cos(\vartheta)\right)}, \tag{2.21a}$$

$$\mu = \frac{R}{-L^2 + iaR\cos(\vartheta)}, \tag{2.21b}$$

$$\tau = \frac{iaR^2\sin(\vartheta)}{\sqrt{2}\left(L^2 - iaR\cos(\vartheta)\right)^2}, \tag{2.21c}$$

$$\pi = -\frac{iaR^2\sin(\vartheta)}{\sqrt{2}\left(a^2R^2\cos^2(\vartheta) + L^4\right)}, \tag{2.21d}$$

$$\epsilon = \frac{R^2\left(a^2(-R) - ia\cos(\vartheta)\left(L^2 - MR\right) + L^2M\right)}{2\left(L^2 - iaR\cos(\vartheta)\right)^2\left(L^2 + iaR\cos(\vartheta)\right)}, \tag{2.21e}$$

$$\alpha = \frac{R\cot(\vartheta)}{\sqrt{2}\left(2L^2 + 2iaR\cos(\vartheta)\right)}, \tag{2.21f}$$

$$\beta = \frac{R\left(-L^2\cot(\vartheta) + iaR\sin(\vartheta)\left(\csc^2(\vartheta) + 1\right)\right)}{2\sqrt{2}\left(L^2 - iaR\cos(\vartheta)\right)^2}. \tag{2.21g}$$

These are all regular on the black hole horizon and future null infinity ($R = 0$). Some of the rotation coefficients are singular at the poles $\vartheta = 0, \varpi$.

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