Notes on perturbations of spherically symmetric spacetimes

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Abstract

We start by reviewing the Einstein equations in spherical symmetry. We then write down the perturbed Einstein equations about a spherically symmetric background. This is mostly a review of a covariant framework for the spherical decomposition of tensors [Mar04, MP05, GMG00, MGG01]. These notes are essentially an outgrowth of notes for the paper [RY18]. Please let me know if you find any typos/errors!

Chapter 1

General equations of motion

Our notation generally follows [Wal84]. For a textbook discussion of relativistic fluids, see [RZ13]. We consider the Einstein equations coupled to fluid matter

$$
\begin{aligned}
E^{(g)}_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa T_{\alpha\beta}, && (1.1)\\
\nabla_\alpha T^{\alpha\beta} = 0, && (1.2)\\
E^{(g)} \equiv \nabla_\alpha J^\alpha = 0. && (1.3)
\end{aligned}
$$

Here $T_{\alpha\beta}$ is the stress-energy tensor, and J^α is the fluid current.

We decompose the stress-energy tensor in terms of the fluid velocity vector u^α, which is a unit timelike vector ($u^\alpha u_\alpha = -1$):

$$
\begin{aligned}
T^{\alpha\beta} &= \mathcal{E}u^\alpha u^\beta + \mathcal{P}\Delta^{\alpha\beta} + \left(\mathcal{Q}^\alpha u^\beta + \mathcal{Q}^\beta u^\alpha\right) + \mathcal{T}^{\alpha\beta}, && (1.4)\\
J^\alpha &= \mathcal{N}u^\alpha + \mathcal{J}^\alpha, && (1.5)
\end{aligned}
$$

where $\mathcal{E}$, $\mathcal{P}$, and $\mathcal{N}$ are scalars, $\mathcal{Q}^\alpha$, $\mathcal{J}^\alpha$ are vectors transverse to u^α (for example $u_\alpha\mathcal{Q}^\alpha = 0$), and $\mathcal{T}^{\alpha\beta}$ is a symmetric transverse-traceless tensor with respect to u^α (that is $u_\alpha\mathcal{T}^{\alpha\beta} = \mathcal{T}^\alpha{}_\alpha = 0$, and

$$
\Delta^{\alpha\beta} \equiv g^{\alpha\beta} + u^\alpha u^\beta, \tag{1.6}
$$

projects onto the space transverse to u^α. More specifically, for a d dimensional spacetime (we work in $d = 4$ spacetime dimensions) we have

$$
\begin{aligned}
\mathcal{E} &\equiv u_\alpha u_\beta T^{\alpha\beta}, && (1.7a)\\
\mathcal{P} &\equiv \frac{1}{d-1}\Delta_{\alpha\beta}T^{\alpha\beta}, && (1.7b)\\
\mathcal{Q}_\alpha &\equiv -\Delta_{\alpha\beta}u_\gamma T^{\beta\gamma}, && (1.7c)\\
\mathcal{N} &\equiv -u_\gamma J^\gamma, && (1.7d)\\
\mathcal{J}_\alpha &\equiv \Delta_{\alpha\beta}J^\beta, && (1.7e)
\end{aligned}
$$

$$\mathcal{T}^{\alpha\beta} \equiv T^{\langle\alpha\beta\rangle}, \tag{1.7f}$$

where the angle brackets of a tensor is defined to be the symmetric transverse-traceless part of the tensor

$$X^{\langle\alpha\beta\rangle} \equiv \frac{1}{2}\left(\Delta^{\alpha\gamma}\Delta^{\beta\delta}\left(X_{\gamma\delta}+X_{\delta\gamma}\right)-\frac{2}{d-1}\Delta^{\alpha\beta}\Delta^{\gamma\delta}X_{\gamma\delta}\right). \tag{1.8}$$

So far we have only given a general decomposition of the stress-energy tensor with respect to a timelike unit vector u^α. Specifying a specific fluid theory requires specifying *constitutive relations* for the quantities $\mathcal{E}$, ..., $\mathcal{T}^{\alpha\beta}$. It's worth noting that the trace of the stress-energy tensor is

$$T = -\mathcal{E} + 3\mathcal{P}, \tag{1.9}$$

that is, the heat flux and shear do not contribute to the trace. The conservation equation $\nabla_\alpha T^{\alpha\beta}=0$ can be split into a part parallel to u^β and perpendicular to u^β (the relativistic generalizations of the continuity and Euler-Navier-Stokes equations):

$$u^\alpha\nabla_\alpha\mathcal{E} + \left(\mathcal{E}+\mathcal{P}\right)\nabla_\alpha u^\alpha + \nabla_\alpha\mathcal{Q}^\alpha - u_\gamma u^\alpha\nabla_\alpha\mathcal{Q}^\gamma - u_\gamma\nabla_\alpha\mathcal{T}^{\alpha\gamma} = 0, \tag{1.10}$$

$$\begin{aligned}\left(\mathcal{E}+\mathcal{P}\right)u^\alpha\nabla_\alpha u^\beta + \mathcal{Q}^\alpha\nabla_\alpha u^\beta + \mathcal{Q}^\beta\nabla_\alpha u^\alpha + \Delta^{\alpha\beta}\nabla_\alpha\mathcal{P}& \\ +\Delta^\beta{}_\gamma u^\alpha\nabla_\alpha\mathcal{Q}^\gamma + \Delta^\beta{}_\gamma\nabla_\alpha\mathcal{T}^{\alpha\gamma} &= 0.\end{aligned} \tag{1.11}$$

The conservation of the current, $\nabla_\alpha J^\alpha = 0$, can be written as

$$u^\alpha\nabla_\alpha\mathcal{N} + \mathcal{N}\nabla_\alpha u^\alpha + \nabla_\alpha\mathcal{J}^\alpha = 0. \tag{1.12}$$

We consider perturbations of non-rotating neutron star solutions, that is perturbations of the Einstein-fluid system:

$$\delta\left(R_{\alpha\beta} - \kappa\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)\right) = 0, \tag{1.13}$$

$$\delta\left(\nabla_\alpha T^{\alpha\beta}\right) = 0, \tag{1.14}$$

$$\delta\left(\nabla_\alpha J^\alpha\right) = 0. \tag{1.15}$$

As we are perturbing about a spherically symmetric background, we can decompose linear perturbations according to how they transform under rotations (irreducible components of the rotation group). We consider perturbations of the metric $\delta g_{\alpha\beta}$, and Eulerian perturbations δu^α of the fluid velocity.

1.1 Perturbation of the Ricci tensor

We start with the well-known identities [Wal84]

$$\delta R^\alpha{}_{\gamma\beta\delta} = \nabla_\beta\delta\Gamma^\alpha_{\delta\gamma} - \nabla_\delta\delta\Gamma^\alpha_{\beta\gamma}, \tag{1.16}$$

$$\delta\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(\nabla_\alpha\delta g_{\delta\beta} + \nabla_\beta\delta g_{\delta\alpha} - \nabla_\delta\delta g_{\alpha\beta}\right). \tag{1.17}$$

We then have

$$\begin{aligned}
\delta R_{\alpha\beta} =& \nabla_\gamma\delta\Gamma^\gamma_{\alpha\beta} - \nabla_\beta\delta\Gamma^\gamma_{\alpha\gamma} \\
=& -\frac{1}{2}g^{\gamma\delta}\nabla_\gamma\nabla_\delta\delta g_{\alpha\beta} + \frac{1}{2}g^{\gamma\delta}\left(\nabla_\alpha\nabla_\gamma\delta g_{\delta\beta} + \nabla_\beta\nabla_\delta\delta g_{\alpha\gamma} - \nabla_\beta\nabla_\alpha\delta g_{\delta\gamma}\right) \\
&+ \frac{1}{2}g^{\gamma\delta}\left[\nabla_\gamma, \nabla_\beta\right]\delta g_{\delta\alpha} + \frac{1}{2}g^{\gamma\delta}\left[\nabla_\gamma, \nabla_\alpha\right]\delta g_{\delta\beta} \\
=& -\frac{1}{2}g^{\gamma\delta}\nabla_\gamma\nabla_\delta\delta g_{\alpha\beta} + \nabla_{(\alpha}v_{\beta)} - R_\alpha{}^\gamma{}_\beta{}^\delta\delta g_{\gamma\delta} + R^\gamma{}_{(\alpha}\delta g_{\beta)\gamma},
\end{aligned} \tag{1.18}$$

where we have defined

$$\boxed{v_\mu \equiv g^{\gamma\delta}\left(\nabla_\gamma\delta g_{\delta\mu} - \frac{1}{2}\nabla_\mu\delta g_{\gamma\delta}\right) = g^{\gamma\delta}\delta\Gamma_{\mu\gamma\delta}.} \tag{1.19}$$

Putting everything together, we have

$$\boxed{\delta R_{\alpha\beta} = -\frac{1}{2}g^{\gamma\delta}\nabla_\gamma\nabla_\delta\delta g_{\alpha\beta} - R_\alpha{}^\gamma{}_\beta{}^\delta\delta g_{\gamma\delta} + R^\gamma{}_{(\alpha}\delta g_{\beta)\gamma} + \nabla_{(\alpha}v_{\beta)}} \tag{1.20}$$

Some authors define the Lichnerowicz wave operator

$$\Box_L\delta g_{\alpha\beta} \equiv -\frac{1}{2}g^{\gamma\delta}\nabla_\gamma\nabla_\delta\delta g_{\alpha\beta} - R_\alpha{}^\gamma{}_\beta{}^\delta\delta g_{\gamma\delta} + R^\gamma{}_{(\alpha}\delta g_{\beta)\gamma}. \tag{1.21}$$

The term $\nabla_{(\alpha}v_{\beta)}$ can be thought as describing pure gauge fluctuations in the linearized Einstein equations. We see the linearized Einstein equations essentially take the form of a system of linear wave equations for the components of $\delta g_{\alpha\beta}$.

Chapter 2

Spherically symmetric spacetime decomposition

2.1 Spherically symmetric spacetime

The metric for a spherically symmetric spacetime can generally be split into the form (for a review see [AV10])

$$\boxed{ds^2 = \alpha_{ab}dx^a dx^b + r^2\Omega_{AB}d\theta^A d\theta^B,} \tag{2.1}$$

where r (the areal radius) depends on the coordinates x^a (e.g. $x^a = (t, r)$ in Schwarzschild-like coordinates). Here Ω_{AB} is the metric for the unit two-sphere. We define the metric compatible derivative for α_{ab} with D_a, and the metric compatible derivative for Ω_{AB} with D_A. The Ricci scalar for α_{ab} is $\mathcal{R}$, and the Ricci scalar for Ω_{AB} is 2. We raise/lower lower case Latin indices with α_{ab}/α^{ab}, and raise/lower upper case Latin indices with Ω_{AB}/Ω^{AB}. We define $r_a \equiv D_a r$, $r_{ab} \equiv D_a D_b r$, and so on. We denote the Lie derivative with respect to a vector ξ^μ with $\mathcal{L}_\xi$.

The nonzero Christoffel symbol components are

$$\Gamma^c_{ab} = {}^{(2)}\Gamma^c_{ab}, \tag{2.2a}$$

$$\Gamma^c_{AB} = -\Omega_{AB} r r^c, \tag{2.2b}$$

$$\Gamma^C_{aB} = \delta^C_B \frac{1}{r} r_a, \tag{2.2c}$$

$$\Gamma^C_{AB} = {}^{(2)}\Gamma^C_{AB}. \tag{2.2d}$$

The nonzero components of the Riemann and Ricci tensors, along with the Ricci scalar, are

$$R_{abcd} = \frac{1}{2}\mathcal{R}\left(\alpha_{ac}\alpha_{bd} - \alpha_{ad}\alpha_{bc}\right), \tag{2.3a}$$

$$R_{aAbB} = -r r_{ab}\Omega_{AB}, \tag{2.3b}$$

$$R_{ABCD} = (1 - r_a r^a) r^2 (\Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}), \tag{2.3c}$$

$$R_{ab} = \frac{1}{2}\mathcal{R}\alpha_{ab} - \frac{2}{r} r_{ab}, \tag{2.3d}$$

$$R_{AB} = (1 - r_a r^a - r r^a_a) \Omega_{AB}, \tag{2.3e}$$

$$R = \mathcal{R} - \frac{4}{r} r^c_c + \frac{2}{r^2}(1 - r_a r^a). \tag{2.3f}$$

The covariant Misner-Sharp mass m is defined by

$$\boxed{1 - \frac{2m}{r} \equiv D_a r D^a r.} \tag{2.4}$$

The spherically symmetric stress-energy tensor can be written as

$$T_{\alpha\beta} dx^\alpha dx^\beta = T_{ab} dx^a dx^b + T_2 r^2 \Omega_{AB} d\theta^A d\theta^B. \tag{2.5}$$

Making use of the fluid-decomposition of the stress-energy tensor, we can also write this as

$$\boxed{\begin{aligned} T_{\alpha\beta} dx^\alpha dx^\beta =& \left(\mathcal{E} u_a u_b + (\mathcal{P} + \mathcal{T}) \Delta_{ab} + \mathcal{Q}_a u_b + \mathcal{Q}_b u_a\right) dx^a dx^b \\ &+ \left(\mathcal{P} - \frac{1}{2}\mathcal{T}\right) r^2 \Omega_{AB} d\theta^A d\theta^B, \end{aligned}} \tag{2.6}$$

where $\Delta_{ab} \equiv u_a u_b + \alpha_{ab}$. Generally we can include a shear term in spherical symmetry, although it is not present in perfect fluids. With this, the spherical decomposition of the spherically symmetric Einstein equations are

$$\frac{2}{r}(\alpha_{ab} r^c_c - r_{ab}) - \frac{1}{r^2}(1 - r_c r^c) \alpha_{ab} = \kappa \left((\mathcal{E} + \mathcal{P} + \mathcal{T}) u_a u_b + \mathcal{P}\alpha_{ab} + \mathcal{Q}_a u_b + \mathcal{Q}_a u_b\right) \tag{2.7}$$

$$\frac{1}{r} r^c_c - \frac{1}{2}\mathcal{R} = \kappa \left(\mathcal{P} - \frac{1}{2}\mathcal{T}\right). \tag{2.8}$$

The spherical decomposition of the fluid equations are

$$u^a D_a \mathcal{E} + (\mathcal{E} + \mathcal{P}) \frac{1}{r^2} D_a \left(r^2 u^a\right) + \frac{1}{r^2} D_a \left(r^2 \mathcal{Q}^a\right) + u_b u^a D_a \mathcal{Q}^b = 0, \tag{2.9}$$

$$u^a D_a u^b + Q^a D_a u^b + Q^b \frac{1}{r^2} D_a \left(r^2 u^a\right) + \Delta^{ab} D_a \mathcal{P} + \Delta^b_c u^a D_a \mathcal{Q}^c = 0. \tag{2.10}$$

The spherical decomposition of the conservation of the current is

$$u^a D_a \mathcal{N} + \mathcal{N} \frac{1}{r^2} D_a \left(r^2 u^a\right) + \frac{1}{r^2} D_a \left(r^2 \mathcal{J}^a\right) = 0. \tag{2.11}$$

2.2 Perturbation of a spherically symmetric spacetime

Following [MP05], we consider linear perturbations of a spherically symmetric metric

$$ds^2 = (\alpha_{ab} + p_{ab})\, dx^a dx^b + 2p_{aA} dx^a d\theta^A + \left(r^2 \Omega_{AB} + p_{AB}\right) d\theta^A d\theta^B. \tag{2.12}$$

The inverse metric to linear order is

$$g^{ab} = \alpha^{ab} - p^{ab}, \tag{2.13a}$$

$$g^{aB} = -\frac{1}{r^2} p^{aB}, \tag{2.13b}$$

$$g^{AB} = \frac{1}{r^2} \Omega^{AB} - \frac{1}{r^4} p^{AB}. \tag{2.13c}$$

The perturbations decomposed with respect to irreducible representations of the rotation group are

$$p_{ab} = \sum_{\ell,m} [h_\ell^m]_{ab}\, Y_\ell^m, \tag{2.14a}$$

$$p_{aA} = \sum_{\ell,m} \left([j_\ell^m]_a\, [E_\ell^m]_A + [h_\ell^m]_a\, [S_\ell^m]_A\right), \tag{2.14b}$$

$$p_{AB} = \sum_{\ell,m} \left(r^2\, [k_\ell^m]\, \Omega_{AB} Y_\ell^m + r^2\, [g_\ell^m]\, [Z_\ell^m]_{AB} + [h_\ell^m]_2\, [S_\ell^m]_{AB}\right). \tag{2.14c}$$

We next review how to construct gauge-invariant linear perturbations in the spherical harmonic decomposition [MP05]. Consider linear gauge transformations

$$\begin{aligned} g'_{\alpha\beta} &= g_{\alpha\beta} - \mathcal{L}_\Xi g_{\alpha\beta} \\ &= g_{\alpha\beta} - \nabla_\alpha \Xi_\beta - \nabla_\beta \Xi_\alpha. \end{aligned} \tag{2.15}$$

We decompose the gauge transformation vector as

$$\Xi_a = \sum_{\ell,m} [\xi_\ell^m]_a\, Y_\ell^m \tag{2.16a}$$

$$\Xi_A = \sum_{\ell,m} \left([\xi_\ell^m]_+\, [E_\ell^m]_A + [\xi_\ell^m]_-\, [S_\ell^m]_A\right). \tag{2.16b}$$

The components of the linearized metric transform as

$$[h_\ell^m]'_{ab} = [h_\ell^m]_{ab} - D_a\, [\xi_\ell^m]_b - D_b\, [\xi_\ell^m]_a \tag{2.17a}$$

$$[j_\ell^m]'_a = [j_\ell^m]_a - [\xi_\ell^m]_a - D_a\, [\xi_\ell^m]_+ + \frac{2}{r} r_a\, [\xi_\ell^m]_+, \tag{2.17b}$$

$$[k_\ell^m]' = [k_\ell^m] + \frac{\ell\,(\ell+1)}{r^2}\, [\xi_\ell^m]_+ - \frac{2}{r} r^a\, [\xi_\ell^m]_a, \tag{2.17c}$$

$$[g_\ell^m]' = [g_\ell^m] - \frac{2}{r^2}\,[\xi_\ell^m]_+\,, \tag{2.17d}$$

$$[h_\ell^m]_a' = [h_\ell^m]_a - D_a\,[\xi_\ell^m]_- + \frac{2}{r} r_a\,[\xi_\ell^m]_-\,, \tag{2.17e}$$

$$[h_\ell^m]_2' = [h_\ell^m]_2 - 2\,[\xi_\ell^m]_-\,. \tag{2.17f}$$

Gauge-invariant combinations of these variables are

$$\left[\tilde{h}_\ell^m\right]_{ab} \equiv [h_\ell^m]_{ab} - D_a\,[\varepsilon_\ell^m]_b - D_b\,[\varepsilon_\ell^m]_a\,, \tag{2.18}$$

$$\left[\tilde{k}_\ell^m\right] \equiv [k_\ell^m] + \frac{1}{2}\ell\,(\ell+1)\,[g_\ell^m] - \frac{2}{r} r^a\,[\varepsilon_\ell^m]_a\,, \tag{2.19}$$

$$\left[\tilde{h}_\ell^m\right]_a \equiv [h_\ell^m]_a - \frac{1}{2} D_a\,[h_\ell^m]_2 + \frac{1}{r} r_a\,[h_\ell^m]_2\,. \tag{2.20}$$

where

$$[\varepsilon_\ell^m]_a \equiv [j_\ell^m]_a - \frac{1}{2} r^2\,[g_\ell^m]\,. \tag{2.21}$$

In the Regge-Wheeler gauge [RW57, TC67]

$$\boxed{[j_\ell^m]_a = [g_\ell^m] = [h_\ell^m]_2 = 0.} \tag{2.22}$$

In this gauge we see that

$$\boxed{\left[\tilde{h}_\ell^m\right]_{ab} = [h_\ell^m]_{ab}\,, \qquad \left[\tilde{k}_\ell^m\right] = [k_\ell^m]\,, \qquad \left[\tilde{h}_\ell^m\right]_a = [h_\ell^m]_a\,.} \tag{2.23}$$

We can then derive the polar and axial equations of motion in Regge-Wheeler gauge, and then make those equations gauge invariant by promoting $[h_\ell^m]_{ab} \to \left[\tilde{h}_\ell^m\right]_{ab}$, $[k_\ell^m] \to \left[\tilde{k}_\ell^m\right]$, and $[h_\ell^m]_a \to \left[\tilde{h}_\ell^m\right]_a$ [MP05].

Similarly to how we decompose the perturbations of the spacetime metric, we can decompose the perturbations of the stress-energy tensor

$$T^\alpha_\beta dx^\beta \partial_\alpha = \left(T^a_b + P^a_b\right) dx^b \partial_a + P^a_B d\theta^B \partial_a + P^A_b d\theta^b \partial_A + \left(\delta^A_B \mathcal{P} + P^A_B\right) d\theta^B \partial_A, \tag{2.24}$$

and set

$$P^a_b = \sum_{\ell,m} [H_\ell^m]^a_b\, Y_\ell^m, \tag{2.25a}$$

$$P^a_B = \sum_{\ell,m} \left([J_\ell^m]^a\,[E_\ell^m]_B + [H_\ell^m]^a\,[S_\ell^m]_B\right), \tag{2.25b}$$

$$P^A_b = \frac{1}{r^2}\Omega^{AC}\alpha_{bc}P^c_C, \tag{2.25c}$$

$$P^A_B = \sum_{\ell,m} \left([K^m_\ell]\, \delta^A_B Y^m_\ell + [G^m_\ell]\, [Z^m_\ell]^A_B + \frac{1}{r^2} [H^m_\ell]_2\, [S^m_\ell]^A_B \right). \tag{2.25d}$$

Under linear gauge transformations, we have

$$\begin{aligned}(T')^\alpha_\beta =& T^\alpha_\beta - \mathcal{L}_\Xi T^\alpha_\beta \\ =& T^\alpha_\beta - \Xi^\gamma \nabla_\gamma T^\alpha_\beta + T^\gamma_\beta \nabla_\gamma \Xi^\alpha - T^\alpha_\gamma \nabla_\beta \Xi^\gamma.\end{aligned} \tag{2.26}$$

If $T_{\alpha\beta} = 0$ on the background, then the linearized stress-energy tensor perturbations are gauge invariant. That is not generally the case for us, though. In general we have

$$\left([H^m_\ell]'\right)^a_b = [H^m_\ell]^a_b - [\xi^m_\ell]^c\, D_c T^a_b + T^c_b D_c\, [\xi^m_\ell]^a - T^a_c D_b\, [\xi^m_\ell]^c \tag{2.27a}$$

$$\left([J^m_\ell]'\right)^a = [J^m_\ell]^a + \mathcal{P}\, [\xi^m_\ell]^a - T^a_c\, [\xi^m_\ell]^c, \tag{2.27b}$$

$$[K^m_\ell]' = [K^m_\ell] - [\xi^m_\ell]^c\, D_c \mathcal{P}, \tag{2.27c}$$

$$[G^m_\ell]' = [G^m_\ell], \tag{2.27d}$$

$$\left([H^m_\ell]'\right)^a = [H^m_\ell]^a, \tag{2.27e}$$

$$[H^m_\ell]'_2 = [H^m_\ell]_2. \tag{2.27f}$$

2.2.1 Perturbation of the Christoffel symbols

To compute the linearized equations of motion, we need the perturbed Christoffel symbol components. Using standard formulas [Wal84], we have

$$\begin{aligned}\delta\Gamma^\gamma_{\alpha\beta} =& \frac{1}{2} g^{\gamma\delta} \left(\nabla_\alpha \delta g_{\delta\beta} + \nabla_\beta \delta g_{\delta\alpha} - \nabla_\delta \delta g_{\alpha\beta}\right) \\ =& \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha \delta g_{\delta\beta} + \partial_\beta \delta g_{\delta\alpha} - \partial_\delta \delta g_{\alpha\beta}\right) - g^{\gamma\delta} \Gamma^\rho_{\alpha\beta} \delta g_{\delta\rho}.\end{aligned} \tag{2.28}$$

We then have

$$\begin{aligned}\delta\Gamma^c_{ab} =& \frac{1}{2} \alpha^{cd} \left(\partial_a p_{db} + \partial_b p_{da} - \partial_d p_{ab}\right) - \alpha^{cd} \Gamma^\rho_{ab} p_{d\rho} \\ =& C^c_{ab},\end{aligned} \tag{2.29}$$

$$\begin{aligned}\delta\Gamma^C_{ab} =& \frac{1}{2} \frac{1}{r^2} \Omega^{CD} \left(\partial_a p_{Db} + \partial_b p_{Da} - \partial_D p_{ab}\right) - \frac{1}{r^2} \Omega^{CD} \Gamma^\rho_{ab} p_{D\rho} \\ =& \frac{1}{2} \frac{1}{r^2} \left(D_a p^C_b + D_b p^C_a - D^C p_{ab}\right),\end{aligned} \tag{2.30}$$

$$\begin{aligned}\delta\Gamma^c_{Ab} =& \frac{1}{2} \alpha^{cd} \left(\partial_A p_{db} + \partial_b p_{dA} - \partial_d p_{Ab}\right) - \alpha^{cd} \Gamma^\rho_{Ab} p_{d\rho} \\ =& \frac{1}{2} \left(D_A p^c_b + D_b p^c_A - D^c p_{bA}\right) - \left(\frac{1}{r} D_b r\right) p^c_A,\end{aligned} \tag{2.31}$$

$$\delta\Gamma^c_{AB} = \frac{1}{2} \alpha^{cd} \left(\partial_A p_{dB} + \partial_B p_{dA} - \partial_d p_{AB}\right) - \alpha^{cd} \Gamma^\rho_{AB} p_{d\rho}$$

$$
\begin{aligned}
&=\frac{1}{2}\left(D_A p^c_B + D_B p^c_A - D^c p_{AB}\right) + \Omega_{AB}\left(r D_d r\right) p^{cd}, && (2.32)\\
\delta\Gamma^C_{Ab} &=\frac{1}{2}\frac{1}{r^2}\Omega^{CD}\left(\partial_A p_{Db} + \partial_b p_{DA} - \partial_D p_{Ab}\right) - \frac{1}{r^2}\Omega^{CD}\Gamma^\rho_{Ab}p_{D\rho}\\
&=\frac{1}{2}\frac{1}{r^2}\left(D_A p^C_b + D_b p^C_A - D^C p_{Ab}\right) - \left(\frac{1}{r^3}D_b r\right)p^C_A, && (2.33)\\
\delta\Gamma^C_{AB} &=\frac{1}{r^2}\Omega^{CD}\left(\partial_A p_{DB} + \partial_B p_{DA} - \partial_D p_{AB}\right) - \frac{1}{r^2}\Omega^{CD}\Gamma^\rho_{AB}p_{D\rho}\\
&=\frac{1}{r^2}C^C_{AB} + \left(\frac{1}{r}D_p r\right)\Omega_{AB}p^{Cp}, && (2.34)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
C^c_{ab} &\equiv\frac{1}{2}\alpha^{cd}\left(D_a p_{db} + D_b p_{da} - D_d p_{ab}\right)\\
C^C_{AB} &\equiv\frac{1}{2}\Omega^{CD}\left(D_A p_{DB} + D_B p_{DA} - D_D p_{AB}\right). && (2.35)
\end{aligned}
$$

Notice that we raise/lowered the capital Latin indices with the metric Ω^{AB}/Ω_{AB}, without any factors of r.

2.3 Perturbation of the stress-energy tensor and conserved vector about spherical symmetry

We consider linear perturbations about a spherically symmetric metric of the fluid stress-energy tensor. The general equation is

$$
\begin{aligned}
\delta\left(\nabla_\gamma T^\gamma_\alpha\right) &=\delta\left(\partial_\gamma T^\gamma_\alpha + \Gamma^\gamma_{\gamma\beta}T^\beta_\alpha - \Gamma^\beta_{\gamma\alpha}T^\gamma_\beta\right)\\
&=\partial_\gamma\delta T^\gamma_\alpha + \delta\Gamma^\gamma_{\gamma\beta}T^\beta_\alpha + \Gamma^\gamma_{\gamma\beta}\delta T^\beta_\alpha - \delta\Gamma^\beta_{\gamma\alpha}T^\gamma_\beta - \Gamma^\beta_{\gamma\alpha}\delta T^\gamma_\beta. && (2.36)
\end{aligned}
$$

The different components are

$$
\begin{aligned}
\delta\left(\nabla_\gamma T^\gamma_a\right) =&\partial_c P^c_a + \partial_C P^C_a + \left(\Gamma^c_{cb} + \Gamma^C_{Cb}\right)P^b_a + \Gamma^C_{CB}P^B_a - \Gamma^b_{ca}P^c_b - \Gamma^B_{Ca}P^C_B\\
&+\left(\delta\Gamma^c_{cb} + \delta\Gamma^C_{Cb}\right)T^b_a - \delta\Gamma^b_{ca}T^c_b - \delta\Gamma^B_{Ca}T^C_B\\
=&D_c P^c_a + D_C P^C_a + \frac{2}{r}r_c P^c_a - \frac{1}{r}r_a P^C_C\\
&- C^b_{ca}T^c_b + \left(C^c_{cb} + \frac{1}{2r^2}D_b p^C_C - \frac{1}{r^3}r_b p^C_C\right)T^b_a - \left(\frac{1}{2r^2}D_a p^C_C - \frac{1}{r^3}r_a p^C_C\right)\mathcal{P} && (2.37a)\\
\delta\left(\nabla_\gamma T^\gamma_A\right) =&\partial_c P^c_A + \partial_C P^C_A + \left(\Gamma^c_{cb} + \Gamma^C_{Cb}\right)P^b_A + \Gamma^C_{CB}P^B_A - \Gamma^B_{CA}P^C_B\\
&+\left(\delta\Gamma^c_{cB} + \delta\Gamma^C_{CB}\right)T^B_A - \delta\Gamma^b_{cA}T^c_b - \delta\Gamma^B_{CA}T^C_B\\
=&D_c P^c_A + D_C P^C_A + \frac{2}{r}r_c P^c_A\\
&+\left(\frac{1}{2}D_A p^c_c - \frac{1}{r}r_c p^c_A\right)\mathcal{P} - \left(\frac{1}{2}D_A p^b_c - \frac{1}{r}r_c p^b_A\right)T^c_b. && (2.37b)
\end{aligned}
$$

2.4 Perturbation of the Ricci tensor in spherical symmetry

We will use the formula

$$\delta R_{\alpha\beta} = -\frac{1}{2} g^{\gamma\delta} \nabla_\gamma \nabla_\delta \delta g_{\alpha\beta} - R_\alpha{}^\gamma{}_\beta{}^\delta \delta g_{\gamma\delta} + R^\gamma{}_{(\alpha} \delta g_{\beta)\gamma} + \nabla_{(\alpha} v_{\beta)} \tag{2.38}$$

where,

$$v_\alpha \equiv g^{\gamma\delta} \nabla_\gamma g_{\alpha\delta} - \frac{1}{2} \nabla_\alpha \left(g^{\rho\sigma} \delta g_{\rho\sigma} \right) . \tag{2.39}$$

We first compute the spherical decomposition of the covariant derivatives of the perturbation of the metric tensor [Mar04]. The first covariant derivatives follow from

$$\nabla_\gamma p_{\alpha\beta} = \partial_\gamma p_{\alpha\beta} - \Gamma^\delta_{\gamma\alpha} p_{\delta\beta} - \Gamma^\delta_{\gamma\beta} p_{\delta\alpha}. \tag{2.40}$$

Using Eq. (2.2), we then have

$$\begin{aligned}
\nabla_c p_{ab} =& \partial_c p_b - \Gamma^d_{ca} p_{db} - \Gamma^d_{cb} p_{da} \\
=& D_c p_{ab}, && (2.41a)\\
\nabla_C p_{ab} =& \partial_C p_{ab} - \Gamma^D_{Ca} p_{Db} - \Gamma^D_{Cb} p_{Da} \\
=& D_C p_{ab} - \frac{2}{r} r_{(a} p_{b)C}, && (2.41b)\\
\nabla_c p_{aB} =& \partial_c p_{aB} - \Gamma^d_{ca} p_{dB} - \Gamma^D_{cB} p_{Da} \\
=& D_c p_{aB} - \frac{1}{r} r_c p_{aB}, && (2.41c)\\
\nabla_C p_{aB} =& \partial_C p_{aB} - \Gamma^D_{Ca} p_{DB} - \Gamma^D_{CB} p_{Da} - \Gamma^d_{CB} p_{da} \\
=& D_C p_{aB} - \frac{1}{r} r_a p_{BC} + r r^d \Omega_{BC} p_{da}, && (2.41d)\\
\nabla_c p_{AB} =& \partial_c p_{AB} - \Gamma^D_{cA} p_{DB} - \Gamma^D_{cB} p_{DA} \\
=& D_c p_{AB} - \frac{2}{r} r_c p_{AB}, && (2.41e)\\
\nabla_C p_{AB} =& \partial_C p_{AB} - \Gamma^D_{CA} p_{DB} - \Gamma^d_{CA} p_{dB} - \Gamma^D_{CB} p_{DA} - \Gamma^d_{CB} p_{dA} \\
=& D_C p_{AB} + 2 r r^d \Omega_{C(A} p_{B)d}. && (2.41f)
\end{aligned}$$

It then follows that

$$\begin{aligned}
v_a =& g^{\gamma\delta} \nabla_\gamma p_{\delta a} - \frac{1}{2} \nabla_a \left(g^{\gamma\delta} p_{\gamma\delta} \right) \\
=& \alpha^{cd} \nabla_c p_{da} + \frac{1}{r^2} \Omega^{CD} \nabla_C p_{Da} - \frac{1}{2} \nabla_a p \\
=& \alpha^{cd} D_c p_{da} + \frac{1}{r^2} \Omega^{CD} \left(D_C p_{Da} - \frac{1}{r} r_a p_{CD} + r r^d \Omega_{CD} p_{ad} \right) - \frac{1}{2} D_a p
\end{aligned}$$

$$
\begin{aligned}
&= D_c p_a{}^c + \frac{1}{r^2} D_C p_a{}^C - \frac{1}{r^3} r_a p_C{}^C + \frac{2}{r} r^d p_{ad} - \frac{1}{2} D_a p && \text{(2.42a)}\\
v_A &= g^{\gamma\delta} \nabla_\gamma p_{\delta A} - \frac{1}{2} \nabla_A \left(g^{\gamma\delta} p_{\gamma\delta} \right) \\
&= \alpha^{cd} \nabla_c p_{dA} + \frac{1}{r^2} \Omega^{CD} \nabla_C p_{DA} - \frac{1}{2} \nabla_A p \\
&= \alpha^{cd} D_c p_{dA} - \frac{1}{r} r^d p_{dA} + \frac{1}{r^2} \Omega^{CD} D_C p_{DA} + \frac{2}{r} r^d \Omega^{CD} \Omega_{C(D} p_{A)d} - \frac{1}{2} D_A p \\
&= D_c p_A{}^c + \frac{1}{r^2} D_C p_A{}^C + \frac{2}{r} r^d p_{Ad} - \frac{1}{2} D_A p && \text{(2.42b)}
\end{aligned}
$$

We also have

$$
\begin{aligned}
\nabla_a v_b &= \partial_a v_b - \Gamma^c_{ab} v_c \\
&= D_a v_b && \text{(2.43a)}\\
\nabla_a v_B &= \partial_a v_B - \Gamma^C_{aB} v_C \\
&= D_a v_B - \frac{1}{r} r_a v_B, && \text{(2.43b)}\\
\nabla_A v_b &= \partial_A v_b - \Gamma^C_{Ab} v_C \\
&= D_A v_b - \frac{1}{r} r_b v_A, && \text{(2.43c)}\\
\nabla_A v_B &= \partial_A v_B - \Gamma^C_{AB} v_C - \Gamma^c_{AB} v_c \\
&= D_A v_B + \Omega_{AB} r r^c v_c. && \text{(2.43d)}
\end{aligned}
$$

We compute the second derivative of the perturbed metric tensor using

$$
\nabla_\delta \nabla_\gamma p_{\alpha\beta} = \partial_\delta \left(\nabla_\gamma p_{\alpha\beta} \right) - \Gamma^\rho_{\delta\gamma} \left(\nabla_\rho p_{\alpha\beta} \right) - \Gamma^\rho_{\delta\alpha} \left(\nabla_\gamma p_{\rho\beta} \right) - \Gamma^\rho_{\delta\beta} \left(\nabla_\gamma p_{\alpha\rho} \right). \tag{2.44}
$$

To compute the perturbation of the Ricci tensor in spherical symmetry, all we have to compute are

$$
\nabla_c \nabla_d p_{\alpha\beta}, \qquad \nabla_C \nabla_D p_{\alpha\beta}. \tag{2.45}
$$

For $\alpha = a, \beta = b$, we have

$$
\begin{aligned}
\nabla_d \nabla_c p_{ab} &= \partial_d \left(\nabla_c p_{ab} \right) - \Gamma^p_{dc} \left(\nabla_p p_{ab} \right) - \Gamma^p_{da} \left(\nabla_c p_{pb} \right) - \Gamma^p_{db} \left(\nabla_c p_{ap} \right) \\
&= D_d D_c p_{ab}, && \text{(2.46a)}\\
\nabla_D \nabla_C p_{ab} &= \partial_D \left(\nabla_C p_{ab} \right) - \Gamma^P_{DC} \left(\nabla_P p_{ab} \right) - \Gamma^p_{DC} \left(\nabla_p p_{ab} \right) - \Gamma^P_{Da} \left(\nabla_C p_{Pb} \right) - \Gamma^P_{Db} \left(\nabla_C p_{aP} \right) \\
&= D_D \left(\nabla_C p_{ab} \right) + r r^p \Omega_{CD} \nabla_p p_{ab} - \frac{2}{r} r_{(a} \nabla_{|C|} p_{b)D} \\
&= D_D \left(D_C p_{ab} - \frac{2}{r} r_{(a} p_{b)C} \right) + r r^p \Omega_{CD} D_p p_{ab} \\
&\quad - \frac{2}{r} r_{(a} \left(D_{|C|} p_{b)D} - \frac{1}{r} r_{b)} p_{CD} + p_{b)p} r r^p \Omega_{CD} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= D_D D_C p_{ab} - \frac{2}{r} r_{(a} D_{|D|} p_{b)C} - \frac{2}{r} r_{(a} D_{|C|} p_{b)D} \\
&\quad + \frac{2}{r^2} r_a r_b p_{CD} + r r^p \Omega_{CD} \left(D_p p_{ab} - \frac{2}{r} r_{(a} p_{b)p} \right)
\end{aligned} \tag{2.46b}
$$

For $\alpha = a, \beta = B$, we have

$$
\begin{aligned}
\nabla_d \nabla_c p_{aB} &= \partial_d \left(\nabla_c p_{aB}\right) - \Gamma^p_{dc} \left(\nabla_p p_{aB}\right) - \Gamma^p_{da} \left(\nabla_c p_{aB}\right) - \Gamma^P_{dB} \left(\nabla_c p_{aP}\right) \\
&= D_d \left(\nabla_c p_{aB}\right) - \frac{1}{r} r_d \nabla_c p_{aB} \\
&= D_d \left(D_c p_{aB} - \frac{1}{r} r_c p_{aB} \right) - \frac{1}{r} r_d \left(D_c p_{aB} - \frac{1}{r} r_c p_{aB} \right) \\
&= D_d D_c p_{aB} + \frac{1}{r^2} r_c r_d p_{aB} - \frac{1}{r} r_{cd} p_{aB} - \frac{1}{r} r_c D_d p_{aB} - \frac{1}{r} r_d D_c p_{aB} + \frac{1}{r^2} r_c r_d p_{aB} \\
&= D_d D_c p_{aB} - \frac{2}{r} r_{(c} D_{d)} p_{aB} + \left(\frac{2}{r^2} r_c r_d - \frac{1}{r} r_{cd} \right) p_{aB},
\end{aligned} \tag{2.47a}
$$

$$
\begin{aligned}
\nabla_D \nabla_C p_{aB} &= \partial_D \left(\nabla_C p_{aB}\right) - \Gamma^P_{DC} \left(\nabla_P p_{aB}\right) - \Gamma^p_{DC} \left(\nabla_p p_{aB}\right) - \Gamma^P_{Da} \left(\nabla_C p_{PB}\right) \\
&\quad - \Gamma^P_{DB} \left(\nabla_C p_{aP}\right) - \Gamma^p_{DB} \left(\nabla_C p_{ap}\right) \\
&= D_D \left(\nabla_C p_{aB}\right) - \frac{1}{r} r_a \nabla_C p_{DB} + r r^p \Omega_{CD} \nabla_p p_{aB} + r r^p \Omega_{BD} \nabla_C p_{ap} \\
&= D_D \left(D_C p_{aB} - \frac{1}{r} r_a p_{BC} + r r^d \Omega_{BC} p_{ad} \right) - \frac{1}{r} r_a \left(D_C p_{BD} + 2 r r^d \Omega_{C(B} p_{D)d} \right) \\
&\quad + r r^p \Omega_{CD} \left(D_p p_{aB} - \frac{1}{r} r_p p_{aB} \right) + r r^p \Omega_{BD} \left(D_C p_{ap} - \frac{2}{r} r_{(a} p_{p)C} \right) \\
&= D_D D_C p_{aB} - \frac{2}{r} r_a D_{(C} p_{D)B} + r r^p \left(\Omega_{CD} D_p p_{aB} - \frac{1}{r} r_a \Omega_{BC} p_{pD} \right) \\
&\quad + 2 r r^p \left(\Omega_{B(C} D_{D)} p_{ap} - \frac{1}{r} r_p \Omega_{D(B} p_{C)a} - \frac{1}{r} r_a \Omega_{D(B} p_{C)p} \right).
\end{aligned} \tag{2.47b}
$$

For $\alpha = A, \beta = B$, we have

$$
\begin{aligned}
\nabla_d \nabla_c p_{AB} &= \partial_d \left(\nabla_c p_{AB}\right) - \Gamma^p_{dc} \left(\nabla_p p_{AB}\right) - \Gamma^P_{dA} \left(\nabla_c p_{PB}\right) - \Gamma^P_{dB} \left(\nabla_c p_{PA}\right) \\
&= D_d \left(\nabla_c p_{AB}\right) - \frac{2}{r} r_d \left(\nabla_c p_{AB}\right) \\
&= D_d D_c p_{AB} - \frac{2}{r} r_{cd} p_{AB} + \frac{6}{r^2} r_c r_d p_{AB} - \frac{4}{r} r_{(c} D_{d)} p_{AB},
\end{aligned} \tag{2.48a}
$$

$$
\begin{aligned}
\nabla_D \nabla_C p_{AB} &= \partial_D \left(\nabla_C p_{AB}\right) - \Gamma^P_{DC} \left(\nabla_P p_{AB}\right) - \Gamma^p_{DC} \left(\nabla_p p_{AB}\right) \\
&\quad - \Gamma^P_{DA} \left(\nabla_C p_{PB}\right) - \Gamma^p_{DA} \left(\nabla_C p_{pB}\right) - \Gamma^P_{DB} \left(\nabla_C p_{PA}\right) - \Gamma^p_{DB} \left(\nabla_C p_{pA}\right) \\
&= D_D \left(\nabla_C p_{AB}\right) + \Omega_{CD} r r^p \left(\nabla_p p_{AB}\right) + 2 \Omega_{D(A} r r^p \left(\nabla_{|C|} p_{B)p}\right) \\
&= D_D \left(D_C p_{AB} + 2 r r^d \Omega_{C(A} p_{B)d} \right) \\
&\quad + r r^p \Omega_{CD} \left(D_p p_{AB} - \frac{1}{r} r_p p_{AB} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2rr^p\Omega_{D(A}\left(D_{|C|}p_{B)p} - \frac{1}{r}r_{|p|}p_{B)C} + rr^q\Omega_{B)C}p_{pq}\right) \\
=&D_D D_C p_{AB} + 2rr^p\left(\Omega_{C(A}D_{|D|}p_{B)p} + \Omega_{D(A}D_{|C|}p_{B)p}\right) \\
&- 2r^p r_p\left(\Omega_{CD}p_{AB} + \Omega_{D(A}p_{B)C}\right) + 2r^2r^pr^q\Omega_{D(A}\Omega_{B)C}p_{pq} + rr^p\Omega_{CD}D_p p_{AB}
\end{aligned} \tag{2.48b}
$$

These expressions match those in Martel's PhD thesis [Mar04].

We can now compute the covariant wave operator acting on the perturbed metric

$$
\begin{aligned}
g^{\gamma\delta}\nabla_\gamma\nabla_\delta p_{ab} &= \left(\alpha^{cd}\nabla_c\nabla_d + \frac{1}{r^2}\Omega^{CD}\nabla_C\nabla_D\right)p_{ab} \\
&= \left(D_cD^c + \frac{1}{r^2}D_CD^C\right)p_{ab} - \frac{4}{r^3}r_{(a}D_{|C|}p_{b)}{}^C \\
&\quad + \frac{2}{r}r^c\left(D_cp_{ab} - \frac{2}{r}r_{(a}p_{b)c}\right) + \frac{2}{r^4}r_{(a}r_{b)}\Omega^{CD}p_{CD}
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
g^{\gamma\delta}\nabla_\gamma\nabla_\delta p_{aB} &= \left(\alpha^{cd}\nabla_c\nabla_d + \frac{1}{r^2}\Omega^{CD}\nabla_C\nabla_D\right)p_{aB} \\
&= \left(D_cD^c + \frac{1}{r^2}D_CD^C\right)p_{aB} - \frac{1}{r}\left(\frac{1}{r}r_cr^c + r_c{}^c\right)p_{aB} - \frac{4}{r^2}r_ar^cp_{Bc} \\
&\quad - \frac{2}{r^3}r_aD_Cp_B{}^C + \frac{2}{r}r^cD_Bp_{ac}
\end{aligned} \tag{2.50}
$$

$$
\begin{aligned}
g^{\gamma\delta}\nabla_\gamma\nabla_\delta p_{AB} &= \left(\alpha^{cd}\nabla_c\nabla_d + \frac{1}{r^2}\Omega^{CD}\nabla_C\nabla_D\right)p_{AB} \\
&= \left(D_cD^c + \frac{1}{r^2}D_CD^C\right)p_{AB} - \frac{2}{r}\left(r_c{}^c + r^cD_c\right)p_{AB} \\
&\quad + \frac{4}{r}r^cD_{(A}p_{B)c} + 2\Omega_{AB}r^cr^dp_{cd}.
\end{aligned} \tag{2.51}
$$

We next consider the covariant components of the Riemann tensor contracted with the perturbed metric.

$$
\begin{aligned}
R_a{}^\gamma{}_b{}^\delta p_{\gamma\delta} =& R_a{}^c{}_b{}^d p_{cd} + R_a{}^C{}_b{}^D p_{CD} \\
=& \frac{1}{2}\mathcal{R}\left(\alpha_{ab}\alpha^{cd} - \delta_a^d\delta_b^c\right)p_{cd} - \frac{1}{r^3}\left(r_{ab}\right)\Omega^{CD}p_{CD}
\end{aligned} \tag{2.52a}
$$

$$
\begin{aligned}
R_a{}^\gamma{}_B{}^\delta p_{\gamma\delta} =& R_a{}^C{}_B{}^d p_{Cd} \\
=& \left(\frac{1}{r}r_a^d\right)p_{Bd}
\end{aligned} \tag{2.52b}
$$

$$
\begin{aligned}
R_A{}^\gamma{}_B{}^\delta p_{\gamma\delta} =& R_A{}^c{}_B{}^d p_{cd} + R_A{}^C{}_B{}^D p_{CD} \\
=& -\Omega_{AB}\left(rr^{cd}\right)p_{cd} + \frac{1}{r^2}\left(1 - r_ar^a\right)\left(\Omega_{AB}\Omega^{CD} - \delta_A^D\delta_B^C\right)p_{CD}.
\end{aligned} \tag{2.52c}
$$

Finally, we consider the covariant components of the Ricci tensor contracted with the perturbed metric

$$\begin{aligned}
R^{\gamma}{}_{(a}p_{b)\gamma} =& R^{c}{}_{(a}p_{b)c} \\
=& \frac{1}{2}\mathcal{R}p_{ab} - \frac{2}{r}r^{c}_{(a}p_{b)c} \qquad (2.53a)\\
R^{\gamma}{}_{(a}p_{B)\gamma} =& \frac{1}{2}\left(R^{c}{}_{a}p_{Bc} + R^{C}{}_{B}p_{aC}\right) \\
=& -\frac{1}{r}r^{c}_{a}p_{Bc} + \left(\frac{1}{4}\mathcal{R} + \frac{1}{2r^{2}}\left(1 - r_{c}r^{c} - rr^{c}_{c}\right)\right)p_{aB} \qquad (2.53b)\\
R^{\gamma}{}_{(A}p_{B)\gamma} =& R^{C}{}_{(A}p_{B)C} \\
=& \frac{1}{r^{2}}\left(1 - r_{c}r^{c} - rr^{c}_{c}\right)p_{AB}. \qquad (2.53c)
\end{aligned}$$

Alternatively, we could make use of the Einstein equations are write this as

$$\begin{aligned}
R^{\gamma}{}_{(a}p_{b)\gamma} =& \kappa\hat{T}^{c}{}_{(a}p_{b)c} \qquad (2.54a)\\
R^{\gamma}{}_{(a}p_{B)\gamma} =& \frac{\kappa}{2}\left(\hat{T}^{c}{}_{a}p_{Bc} + \hat{T}^{C}{}_{B}p_{aC}\right) \\
=& \frac{\kappa}{2}\left(\hat{T}^{c}{}_{a}p_{Bc} - \left(\mathcal{E} - \mathcal{P}\right)p_{aB}\right) \qquad (2.54b)\\
R^{\gamma}{}_{(A}p_{B)\gamma} =& \frac{\kappa}{2}\left(\mathcal{E} - \mathcal{P}\right)p_{AB}. \qquad (2.54c)
\end{aligned}$$

Chapter 3

Axial and polar spherical harmonic decomposition of the Einstein equations

3.1 Axial perturbations

We only need to consider the components δR_{aB} and δR_{AB}. As we reviewed in Sec. 2.2, as first shown in [MP05], we can work in the Regge-Wheeler gauge, and then promote the variables to gauge-invariant ones at the end. We also drop the ℓ, m labels to make the equations less cluttered. We use $\doteq$ to indicate that we are dropping all terms that are zero for an axial perturbation in Regge-Wheeler gauge. The only nonzero component of the metric perturbation then is

$$p_{aB} \doteq h_a S_B. \tag{3.1}$$

3.1.1 Computing the aB component of the Ricci tensor

We first look at

$$\begin{aligned} g^{\gamma\delta}\nabla_\gamma\nabla_\delta p_{aB} &\doteq \left(D_c D^c + \frac{1}{r^2}D_C D^C\right) S_B h_a - \frac{1}{r}\left(\frac{1}{r}r_c r^c + r^c_c\right) S_B h_a - \frac{4}{r^2} r_a r^c h_c S_B \\ &= \left(\left(D_c D^c + \frac{1}{r^2}\left(1 - \ell\left(\ell+1\right)\right) - \frac{1}{r^2}r_c r^c - \frac{1}{r}r_c{}^c\right) h_a - \frac{4}{r^2} r_a r_c h^c\right) S_B. \end{aligned} \tag{3.2}$$

We next look at

$$\begin{aligned} \nabla_a v_B + \nabla_B v_a &\doteq \left(D_a - \frac{2}{r}r_a\right) v_B \\ &\doteq \left(D_a - \frac{2}{r}r_a\right)\left(D_c h^c + \frac{2}{r}r_c h^c\right) S_B \\ &= \left(D_a D_c h^c + \frac{2}{r}r_c D_a h^c - \frac{2}{r}r_a D_c h^c + \frac{2}{r}r_{ac}h^c - \frac{6}{r^2}r_a r_c h^c\right) S_B. \end{aligned} \tag{3.3}$$

For the Riemann and Ricci tensor components, we have

$$R_a{}^\gamma{}_B{}^\delta p_{\gamma\delta} = \left(\frac{1}{r} r_{ac}\right) h^c S_B \tag{3.4}$$

$$R^\gamma{}_{(a} p_{B)\gamma} = \frac{1}{2}\left(\kappa \hat{T}_{ac} h^c + \frac{1}{r^2}\left(1 - r_c r^c - r r^c_{c}\right) h_a\right) S_B. \tag{3.5}$$

We have substituted the trace-reverse of the stress-energy tensor $\hat{T}_{ab}$ for R_{ab}, while directly writing out the expression for R_{AB}, so our formulas will match those in [MP05][1]. Using Eq. (1.20), and promoting everything to gauge-invariant quantities, we are left with

$$\boxed{\delta R_{aB} = \frac{1}{2}\left(-\left(D_c D^c - \frac{1}{r^2}\ell\left(\ell+1\right)\right)\tilde{h}_a + D_a D_c \tilde{h}^c + \frac{4}{r} r_{[c} D_{a]} \tilde{h}^c - \frac{2}{r^2} r_a r_c \tilde{h}^c + \kappa \hat{T}_{ac} \tilde{h}^c\right) S_B.} \tag{3.6}$$

We can now write down the tensor equations of motion,

$$\delta R_{aB} \doteq \kappa \left(\delta T_{aB} - \frac{1}{2}\delta g_{aB} T\right). \tag{3.7}$$

3.1.2 Computing the AB component of the Ricci tensor

We first look at

$$\begin{aligned} g^{\gamma\delta}\nabla_\gamma \nabla_\delta p_{AB} &\doteq \frac{4}{r} r_c \Omega^{CD} \Omega_{C(A} D_{|D|} S_{B)} h^c \\ &= \frac{4}{r} r_c h^c S_{AB}. \end{aligned} \tag{3.8}$$

We next look at

$$\begin{aligned} \nabla_A v_B + \nabla_B v_A &\doteq D_A v_B + D_B v_A \\ &\doteq \left(D_A S_B + D_B S_A\right)\left(D_c h^c + \frac{2}{r} r_c h^c\right) \\ &= 2\left(D_c h^c + \frac{2}{r} r_c h^c\right) S_{AB} \end{aligned} \tag{3.9}$$

where, $S_{AB} \equiv D_{(A} S_{B)}$. The other terms $R_A{}^\gamma{}_B{}^\gamma p_{\gamma\delta}$ and $R^\gamma{}_{(A} p_{B)\gamma}$ are zero for axial perturbations in the Regge-Wheeler gauge. Using Eq. (1.20), and promoting everything to gauge-invariant quantities, we have

$$\boxed{\delta R_{AB} = D_c \tilde{h}^c S_{AB}.} \tag{3.10}$$

Note that the axial perturbation of the AB component of the Einstein and Ricci tensors are the same in the Regge-Wheeler gauge

$$\delta G_{AB} \doteq \delta R_{AB}. \tag{3.11}$$

This is why our expression Eq. (3.10) matches Eq (5.9) of [MP05].

[1] Their Eq. 5.8, although note that those authors assume the background is vacuum so $R_{\mu\nu} = 0$.

3.1.3 Computing the a component of the Bianchi identity

There are no axial perturbations of this component.

3.1.4 Computing the A component of the Bianchi identity

We next consider the divergence of the stress-energy tensor (see Sec. (2.3)). We first look at

$$\begin{aligned} D_c P^c_A + \frac{2}{r} r_c P^c_A + D_C P^C_A \doteq & S^A \left(D_c H^c + \frac{2}{r} r_c H^c \right) + \frac{1}{r^2} D_C S^C_A H_2 \\ = & S^A \left(\frac{1}{r^2} D_c \left(r^2 H^c \right) + \frac{1}{r^2} \left(1 - \frac{1}{2} \ell \left(\ell + 1 \right) \right) H_2 \right) . \end{aligned} \tag{3.12}$$

Next, looking at the metric perturbations, we have

$$\left(\frac{1}{2} D_A p^c_c - \frac{1}{r} r_c p^c_A \right) \mathcal{P} - \left(\frac{1}{2} D_A p^c_b - \frac{1}{r} r_b p^c_A \right) T^b_c \doteq S^A \left(-\frac{1}{r} r_c h^c \mathcal{P} + \frac{1}{r} r_b h^c T^b_c \right) . \tag{3.13}$$

Promoting h_a to its gauge invariant counterpart $\tilde{h}_a$, we conclude that the axial matter equations of motion are

$$\boxed{\frac{1}{r^2} D_c \left(r^2 H^c \right) - \frac{\left(\ell - 1 \right) \left(\ell + 2 \right)}{2r^2} H_2 + \frac{1}{r} r_c \tilde{h}^b T^c_b - \frac{1}{r} r_c \tilde{h}^c \mathcal{P} = 0.} \tag{3.14}$$

3.2 Polar spherical harmonic decomposition of the Einstein equations

We need to consider the components δR_{ab}, δR_{aB}, and δR_{AB}. As we review in Sec. 2.2, we can work in the Regge-Wheeler gauge, and promote all variables to their gauge-invariant counterparts at the end of the calculation. We now use $\doteq$ to indicate that we only keep terms that are nonzero in a polar decomposition in Regge-Wheeler gauge. We work in a spherical harmonic basis and drop the ℓ, m labels to make the expressions less cluttered. The only nonzero components of the metric perturbation are

$$p_{ab} \doteq h_{ab} Y, \tag{3.15a}$$

$$p_{AB} \doteq r^2 k \Omega_{AB} Y. \tag{3.15b}$$

3.2.1 Computing the ab components of the Ricci tensor

We first look at

$$g^{\gamma\delta} \nabla_\gamma \nabla_\delta p_{ab} \doteq \left(D_c D^c + \frac{1}{r^2} D_C D^C \right) h_{ab} Y + \frac{2}{r} r^c \left(D_c h_{ab} - \frac{2}{r} r_{(a} h_{b)c} \right) Y + \frac{4}{r^2} r_{(a} r_{b)} k Y$$

$$= \left(\left(D_c D^c - \frac{\ell(\ell+1)}{r^2} \right) h_{ab} + \frac{2}{r} r^c \left(D_c h_{ab} - \frac{2}{r} r_{(a} h_{b)c} \right) + \frac{4}{r^2} r_{(a} r_{b)} k \right) Y. \quad (3.16)$$

We next look at (we can symmetrize later)

$$\begin{aligned}
\nabla_a v_b \doteq & D_a \left(D_c h_b{}^c - \frac{2}{r} r_b k + \frac{2}{r} r^c h_{bc} - \frac{1}{2} D_b (h + 2k) \right) Y \\
= & \Bigg(D_a D_c h_b{}^c + \frac{2}{r^2} r_a r_b k + \frac{1}{r} r^c \left(2 D_{(a} h_{b)c} - D_c h_{ab} \right) \\
& - \frac{2}{r^2} r_a r^c h_{bc} + \frac{2}{r} \left(r^c_a h_{bc} + r^c D_a h_{bc} \right) - D_a D_b \left(\frac{1}{2} h + k \right) \Bigg) Y. \quad (3.17)
\end{aligned}$$

For the Riemann and Ricci tensor components, we have

$$R_a{}^\gamma{}_b{}^\delta p_{\gamma\delta} \doteq \left(\frac{1}{2} \mathcal{R} \left(\alpha_{ab} \alpha^{cd} - \delta^c_a \delta^d_b \right) h_{cd} - \frac{2}{r} r_{ab} k \right) Y, \quad (3.18)$$

$$R^\gamma{}_{(a} p_{b)\gamma} \doteq \left(\frac{1}{2} \mathcal{R} h_{ab} - \frac{2}{r} r^c_{(a} h_{b)c} \right) Y. \quad (3.19)$$

Using Eq. (1.20), and promoting everything to gauge-invariant quantities, we end up with

$$\boxed{\begin{aligned}
\delta R_{ab} \doteq & \Bigg(-\frac{1}{2} \left(D_c D^c - \frac{\ell(\ell+1)}{r^2} \right) \tilde{h}_{ab} + D_{(a} D^c \tilde{h}_{b)c} - D_a D_b \left(\frac{1}{2} \tilde{h} + \tilde{k} \right) \\
& + \frac{2}{r} r^c \left(D_c \tilde{h}_{ab} + D_{(a} \tilde{h}_{b)c} \right) - \frac{2}{r} r_{(a} D_{b)} \tilde{k} + \frac{1}{2} \mathcal{R} \left(3 \tilde{h}_{ab} - \alpha_{ab} \tilde{h} \right) \Bigg) Y.
\end{aligned}} \quad (3.20)$$

3.2.2 Computing the aB component of the Ricci tensor

We first look at

$$g^{\gamma\delta} \nabla_\gamma \nabla_\delta p_{aB} \doteq \left(-\frac{2}{r} r_a k + \frac{2}{r} r^c h_{ac} \right) E_B. \quad (3.21)$$

We next look at

$$\begin{aligned}
\nabla_a v_B + \nabla_B v_a = & D_a v_B + D_B v_a - \frac{2}{r} r_a v_B \\
\doteq & \left(D_c h_a{}^c - D_a k - D_a h + \frac{2}{r} r_c h_a{}^c + \frac{1}{r} r_a h - \frac{2}{r} r_a k \right) E_B. \quad (3.22)
\end{aligned}$$

The Riemann tensor components are zero

$$R_a{}^\gamma{}_B{}^\delta p_{\gamma\delta} \doteq 0. \quad (3.23)$$

The Ricci tensor components are also zero

$$R^{\gamma}{}_{(a}p_{B)\gamma}\doteq 0. \tag{3.24}$$

Using Eq. (1.20), and promoting to gauge-invariant quantities, we have

$$\boxed{\delta R_{aB}\doteq\frac{1}{2}\left(D_c\tilde{h}_a{}^c-D_a\tilde{h}-D_a\tilde{k}+\frac{1}{r}r_a\tilde{h}\right)E_B.} \tag{3.25}$$

3.2.3 Computing the AB components of the Ricci tensor

We first look at

$$\begin{aligned} g^{\gamma\delta}\nabla_\gamma\nabla_\delta p_{AB}\doteq&\left(D_cD^c+\frac{1}{r^2}D_CD^C\right)\Omega_{AB}r^2kY-\frac{2}{r}\left(r_c^c+r^cD_c\right)\Omega_{AB}r^2kY+2\Omega_{AB}r^cr^dh_{cd}Y\\ =&\left(\left(D_cD^c-\frac{\ell\left(\ell+1\right)}{r^2}\right)k+\frac{2}{r}r^cD_ck-\frac{2}{r^2}r_cr^ck+\frac{2}{r^2}r^cr^dh_{cd}\right)r^2\Omega_{AB}Y. \end{aligned} \tag{3.26}$$

We next look at (we can symmetrize later)

$$\begin{aligned} \nabla_Av_B=&D_Av_B+\Omega_{AB}rr^cv_c\\ =&-\frac{1}{2}hZ_{AB}\\ &+\left(\frac{1}{r}r^cD_dh_c{}^d-\frac{1}{r}r^cD_ch-\frac{2}{r}r^cD_ck+\frac{2}{r^2}r^cr^dh_{cd}-\frac{2}{r^2}r_cr^ck-\frac{\ell\left(\ell+1\right)}{2r^2}h\right)r^2\Omega_{AB}Y. \end{aligned} \tag{3.27}$$

The Riemann tensor components are

$$R_A{}^\gamma{}_B{}^\delta p_{\gamma\delta}\doteq\left(-\frac{1}{r}r^{cd}h_{cd}+\frac{1-r_ar^c}{r^2}k\right)r^2\Omega_{AB}Y. \tag{3.28}$$

The Ricci tensor components are also zero

$$R^{\gamma}{}_{(A}p_{B)\gamma}\doteq\left(\frac{1-r_ar^a-rr_a^a}{r^2}k\right)r^2\Omega_{AB}Y. \tag{3.29}$$

Using Eq. (1.20), and promoting to gauge-invariant quantities, we have

$$\boxed{\begin{aligned} \delta R_{AB}=&-\frac{1}{2}\tilde{h}Z_{AB}\\ &+\Bigg(-\frac{1}{2}\left(D_cD^c-\frac{\ell\left(\ell+1\right)}{r^2}\right)\tilde{k}-\frac{3}{r}r^cD_c\tilde{k}-\left(\frac{1}{r^2}r^cr^c-\frac{1}{r}r_c^c\right)\tilde{k}\\ &+\frac{1}{r}r^cD_d\tilde{h}_c{}^d-\frac{1}{r}r^cD_c\tilde{h}+\frac{2}{r^2}r^cr^d\tilde{h}_{cd}-\frac{\ell\left(\ell+1\right)}{2r^2}\tilde{h}\Bigg)r^2\Omega_{AB}Y. \end{aligned}} \tag{3.30}$$

3.2.4 Computing the a component of the Bianchi identity

We consider the divergence of the stress-energy tensor. We first look at

$$D_c P^c_a + D_C P^C_a + \frac{2}{r} r_c P^c_a - \frac{1}{r} r_a P^C_C = \left(D_c H^c_a + \frac{2}{r} r_c H^c_a - \frac{\ell\left(\ell+1\right)}{r^2} J_a - \frac{2}{r} r_a K \right) Y. \tag{3.31}$$

We next look at the metric terms

$$-C^b_{ca} T^b_b + \left(C^c_{cb} - \frac{1}{r^4} r_b p^C_C \right) T^b_a + \frac{1}{r^3} r_a p^C_C \mathcal{P} = \left(-C^b_{ca} T^b_b + \left(C^c_{cb} - \frac{2}{r^2} r_b k \right) T^b_a + \frac{2}{r} r_a k \mathcal{P} \right) Y. \tag{3.32}$$

Putting everything together, we have

$$\boxed{\frac{1}{r^2} D_c \left(r^2 H^c_a \right) - \frac{\ell\left(\ell+1\right)}{r^2} J_a - \frac{2}{r} r_a K - C^b_{ca} T^b_b + \left(C^c_{cb} - \frac{2}{r^2} r_b k \right) T^b_a + \frac{2}{r} r_a k \mathcal{P} = 0.} \tag{3.33}$$

3.2.5 Computing the A component of the Bianchi identity

We consider the divergence of the stress-energy tensor. We first look at

$$D_c P^c_A + D_C P^C_A + \frac{2}{r} r_c P^c_A = \left(D_c J^c + K - \frac{\left(\ell+2\right)\left(\ell-1\right)}{2} G + \frac{2}{r} r_c J^c \right) E_A. \tag{3.34}$$

We next look at the metric terms

$$\frac{1}{2} D_A p^c_c \mathcal{P} - \frac{1}{2} D_A p^b_c T^c_b = \frac{1}{2} \left(h\mathcal{P} - T^b_c h^c_b \right) E_A. \tag{3.35}$$

Putting everything together, we have

$$\boxed{\frac{1}{r^2} D_c \left(r^2 J^c \right) + K - \frac{\left(\ell+2\right)\left(\ell-1\right)}{2} G + h\mathcal{P} - T^b_c h^c_b = 0.} \tag{3.36}$$

Appendix A

Scalar, vector, and tensor spherical harmonics

We work on the unit two-sphere $\mathbb{S}^2$, with metric Ω_{AB}, Levi-Cevita tensor ε_{AB}, and metric compatible derivative D_A. The Ricci tensor is $R = +2$. Our notation for the spherical harmonics follows that of [NR05].

A.1 Scalar spherical harmonics

The scalar spherical harmonics satisfy

$$\left(\Omega^{AB} D_A D_B + \ell\left(\ell+1\right)\right) Y_\ell^m = 0, \tag{A.1}$$

along with the following orthogonality relation

$$\int d\Omega Y_\ell^m Y_{\ell'}^{m'} = \delta_{\ell\ell'}\delta_{mm'}. \tag{A.2}$$

A.2 Vector spherical harmonics

The polar and axial vector spherical harmonics respectively are

$$[E_\ell^m]_A \equiv D_A Y_\ell^m, \qquad [S_\ell^m]_A \equiv \varepsilon_{BA} D^B Y_\ell^m. \tag{A.3}$$

The vector spherical harmonics satisfy

$$\left(\Omega^{AB} D_A D_B + \left(-1 + \ell\left(\ell+1\right)\right)\right)[V_\ell^m]_C = 0, \tag{A.4}$$

along with the following orthogonality relation

$$\int d\Omega \left[V_\ell^m\right]_A \left[V_{\ell'}^{m'}\right]^A = \ell\left(\ell+1\right)\delta_{\ell\ell'}\delta_{mm'}. \tag{A.5}$$

The divergence of the polar and axial vector spherical harmonics respectively are

$$\begin{aligned} D_A \left[E_\ell^m\right]^A =& D_A D^A Y_\ell^m \\ =& -\ell\left(\ell+1\right) Y_\ell^m. \end{aligned} \tag{A.6}$$

$$\begin{aligned} D_A \left[S_\ell^m\right]^A =& \varepsilon_{BA} D^B D^A Y_\ell^m \\ =& 0. \end{aligned} \tag{A.7}$$

A.3 Tensor spherical harmonics

The polar and axial tensor spherical harmonics respectively are

$$\left[Z_\ell^m\right]_{AB} \equiv D_A D_B Y_\ell^m + \frac{1}{2}\ell\left(\ell+1\right)\Omega_{AB} Y_\ell^m, \qquad \left[S_\ell^m\right]_{AB} \equiv D_{(A}\left[S_\ell^m\right]_{B)}. \tag{A.8}$$

The tensor spherical harmonics satisfy

$$\left(\Omega^{AB} D_A D_B + \left(-2+\ell\left(\ell+1\right)\right)\right)\left[T_\ell^m\right]_{CD} = 0, \tag{A.9}$$

along with the following orthogonality relation

$$\int d\Omega \left[T_\ell^m\right]_{AB} \left[T_{\ell'}^{m'}\right]^{AB} = \frac{1}{2}\left(\ell-1\right)\ell\left(\ell+1\right)\left(\ell+2\right)\delta_{\ell\ell'}\delta_{mm'}. \tag{A.10}$$

The polar and axial tensor spherical harmonics are both traceless

$$\begin{aligned} \Omega^{AB}\left[Z_\ell^m\right]_{AB} =& D_A D^A Y_\ell^m + \ell\left(\ell+1\right) Y_\ell^m \\ =& 0, \end{aligned} \tag{A.11}$$

$$\begin{aligned} \Omega^{AB}\left[S_\ell^m\right]_{AB} =& D_A \left[S_\ell^m\right]^A \\ =& 0. \end{aligned} \tag{A.12}$$

The trace is captured by the scalar spherical harmonic Y_ℓ^m, which is sometimes denoted by [Mar04]

$$\left[U_\ell^m\right]_{AB} \equiv \Omega_{AB} Y_\ell^m. \tag{A.13}$$

The divergence of the polar and axial tensor spherical harmonics respectively are

$$\begin{aligned} D_A \left[Z_\ell^m\right]^{AB} =& D_A D^A D^B Y_\ell^m + \frac{1}{2}\ell\left(\ell+1\right) D^B Y_\ell^m \\ =& D^B D_A D^A Y_\ell^m + R_C^B D^C Y_\ell^m + \frac{1}{2}\ell\left(\ell+1\right) D^B Y_\ell^m \\ =& D^B Y_\ell^m - \frac{1}{2}\ell\left(\ell+1\right) D^B Y_\ell^m \\ =& \left(1-\frac{1}{2}\ell\left(\ell+1\right)\right)\left[E_\ell^m\right]^B, \end{aligned} \tag{A.14}$$

$$
\begin{aligned}
D_A \left[S_\ell^m\right]^{AB} =& \frac{1}{2} D_A \left(D^A \left[S_\ell^m\right]^B + D^B \left[S_\ell^m\right]^A \right) \\
=& \frac{1}{2} \left(\left(1 - \ell\left(\ell + 1\right)\right) \left[S_\ell^m\right]^B + D_B D_A \left[S_\ell^m\right[^A + R_C^B \left[S_\ell^m\right]^C \right) \\
=& \left(1 - \frac{1}{2} \ell\left(\ell + 1\right) \right) \left[S_\ell^m\right]^B . \qquad \text{(A.15)}
\end{aligned}
$$

We have used that $R = 2$ and $R_{AB} = (R/2)\Omega_{AB} = \Omega_{AB}$.

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