Studying black hole ringdown with hyperboloidal methods

Infinity Seminar

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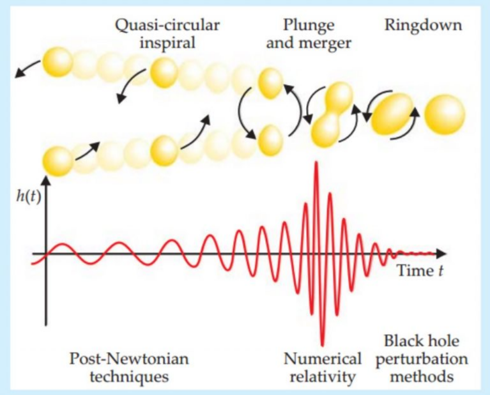
Why study black hole ringdown?

1. Remnant BH: mass M, spin a
2. Model ringdown with quasinormal modes (QNMs):

$$h(t) \sim \sum_i e^{-t/\tau_i} \sin(\omega_i t)$$

3. Black hole perturbation theory

$$\rightarrow \tau_i(M,a),\ \omega_i(M,a)$$



Baumgarte+Shapiro 2010

Black hole spectroscopy: reconstructing the remnant black hole mass/spin from measured quasinormal modes (Detweiler 1980, Dreyer+ 2004, Berti+ 2006, Carullo+ 2019, Isi+ 2021, ...)

What's the problem?

Data analysis:

1. There are many quasinormal modes: which ones to fit to the ringdown?
2. When does ringdown "start"?

Modeling:

1. How well do the quasinormal modes describe the ringdown?
2. Non-mode contributions from nonlinearities (Einstein equations)
3. Non-mode contributions from linear solution

Outline

1. Modeling ringdown with black hole perturbation theory + quasinormal modes
2. Quasinormal modes + boundary conditions
3. Hyperboloidal slicings of Kerr
4. Horizon penetrating, hyperboloidally compactified (HPHC) coordinates and quasinormal modes
    4.1 Open source implementations: https://github.com/JLRipley314
5. Application: measuring quasinormal modes

Modeling black hole ringdown

Final state: Kerr black hole + gravitational waves

Kerr black hole (Boyer-Lindquist coordinates)

$$g_{\alpha\beta}^{(Kerr)} dx^\alpha dx^\beta = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar\sin^2\theta}{\Sigma} dt d\varphi$$
$$- \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right) d\varphi^2$$
$$\Sigma \equiv r^2 + a^2 \cos^2\theta \qquad \Delta \equiv r^2 - 2Ma + a^2.$$

Ringdown: small perturbations about remnant black hole

$$g_{\alpha\beta} = g_{\alpha\beta}^{(Kerr)} + \epsilon g_{\alpha\beta}^{(1)} + \epsilon^2 g_{\alpha\beta}^{(2)} + \mathcal{O}\left(\epsilon^3\right).$$

Linear black hole perturbation theory

Linear perturbation of metric

$$g_{\alpha\beta} = \boxed{g^{(Kerr)}_{\alpha\beta} + \epsilon g^{(1)}_{\alpha\beta}} + \epsilon^2 g^{(2)}_{\alpha\beta} + \mathcal{O}\left(\epsilon^3\right).$$

Linear perturbation of equations of motion

$$R_{\mu\nu}\left[g_{\alpha\beta}\right] = \boxed{R^{(0)}_{\mu\nu}\left[g^{(Kerr)}_{\alpha\beta}\right] + \epsilon R^{(1)}_{\mu\nu}\left[g^{(1)}_{\alpha\beta}\right]} + \epsilon^2 R^{(1)}_{\mu\nu}\left[g^{(2)}_{\alpha\beta}\right] + \epsilon^2 R^{(2)}_{\mu\nu}\left[g^{(1)}_{\alpha\beta}, g^{(1)}_{\gamma\delta}\right] + \mathcal{O}\left(\epsilon^3\right).$$

Look for mode solutions to the linearized Einstein equations

$$R^{(1)}_{\mu\nu}\left[g^{(1)}_{\alpha\beta}\right] = 0, \qquad g^{(1)}_{\mu\nu}\left(t, r, \theta, \phi\right) \sim e^{i\omega t}\hat{g}_{\mu\nu}\left(r, \theta, \phi\right)$$

Equation of motion for linearized curvature perturbation

Teukolsky equation (Teukolsky 1973)

$$\mathcal{T}_{-2}\Psi_4^{(1)} = 0$$

$$\left[\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\right]\partial_t^2\Psi - \Delta^{-2}\partial_r\left(\Delta^{s+1}\partial_r\Psi\right) + \frac{4Mar}{\Delta}\partial_t\partial_\varphi\Psi$$
$$+\frac{a^2}{\Delta}\partial_\varphi^2\Psi - D_{\mathbb{S}^2}^2\Psi - 2s\frac{a\left(r-M\right)}{\Delta}\partial_\varphi\Psi$$
$$-2s\left[\frac{M\left(r^2-a^2\right)}{\Delta} - r - ia\cos\theta\right]\partial_t\Psi = 0.$$

$\Psi \to \Psi_4^{(1)}$ ($s=-2$): complex, linearized **Newman-Penrose** scalar (Newman+Penrose, 1962)

$\Psi_4^{(1)}$ completely describes dynamical part of linearly perturbed metric $g_{\mu\nu}^{(1)}$ (Chrzanowski 1975, Wald 1978)

Teukolsky equation + quasi-normal modes

1. Quasi-normal modes: mode-like solutions to Teukolsky equation

$$\Psi_4^{(1)} = e^{-i\omega t + im\phi} R\left(r\right) S\left(\theta\right).$$

2. Teukolsky equation separates

$$A_1 \frac{d^2 R}{dr} + B_1 \frac{dR}{dr} + C_1 R = \Lambda R,$$
$$A_2 \frac{d^2 S}{d\theta^2} + B_2 \frac{dS}{d\theta} + C_2 S = \Lambda S.$$

3. Quasi-normal modes: solution separated equations + satisfy outgoing (wave) boundary conditions

Boundary conditions (Schwarzschild black holes)

1. Tortoise coordinate

$$r_* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right).$$

   Note: $\lim_{r \to r_+} r_* = -\infty$, and $\lim_{r \to \infty} r_* = +\infty$

2. Ingoing boundary condition at black hole horizon

$$\lim_{r_* \to -\infty} \Psi_4^{(1)} \propto e^{-i\omega(t+r_*)}\left(1 + \cdots\right)$$

3. Outgoing boundary condition at future null infinity

$$\lim_{r_* \to \infty} \Psi_4^{(1)} \propto \frac{1}{r} e^{-i\omega(t-r_*)}\left(1 + \cdots\right)$$

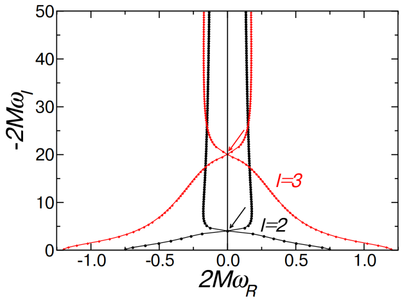
4. Quasinormal modes: ω, R, S that satisfy separated equations + boundary conditions

   - **Quasinormal eigenfunction** (QNE): $\Psi_4^{(1)}$
   - **Quasinormal mode** (QNM): ω

Indexing of QNMs

$$\omega_{[p],\ell,m,n}$$

$[p]$: prograde vs. retrograde
ℓ: angular quantum number
m: azimuthal angular quantum number
n: overtone number



Berti+ 2009

Quasinormal modes in hyperboloidal coordinates

Zenginoğlu Phys.Rev.D 83 (2011) 127502

JLR Class.Quant.Grav. 39 (2022) 14, 145009

https://github.com/JLRipley314/TeukolskyQNMFunctions.jl

Quasinormal modes

$$\psi = Ae^{-i\omega t + im\phi} R\left(r\right) S\left(\theta\right).$$

Decaying mode: $\mathcal{I}\omega < 0 \implies$

$$\psi \sim Ae^{-\mathcal{I}\omega t} \cos\left(\mathcal{R}\omega t\right)\left(1 + \cdots\right).$$

Outgoing boundary condition at future null infinity

$$\lim_{r_* \to \infty} \psi \sim e^{-i\omega(t - r_*)} \frac{A}{r}\left(1 + \cdots\right).$$

$\implies$ blowup as $r \to \infty$

$$\lim_{r_* \to \infty} \psi \sim e^{-\mathcal{I}\omega(t - r_*)} \cos\left(\mathcal{R}\omega\left(t - r_*\right)\right) \frac{A}{r}\left(1 + \cdots\right).$$

Resolution

1. Is this a problem? **No**
2. No “real physics” at $t = const.$, $r \to \infty$
   - Causally disconnected from the interior of the spacetime
   - Blowup is an artefact of looking at a damped infinite wave solution over the entire spacetime

Resolution

1. Transform to outgoing null coordinate

$$u = t - r_*.$$

2. Outgoing boundary condition at future null infinity

$$\lim_{r_* \to \infty} \psi \sim e^{-i\omega u} \frac{A}{r} \left(1 + \cdots\right).$$

3. No more blowup when $u = const.$, $r \to \infty$; wave decays as $u \to \infty$

$$\lim_{r_* \to \infty} \psi \sim e^{-\mathcal{I}\omega u} \cos\left(\mathcal{R}\omega u\right) \frac{A}{r} \left(1 + \cdots\right).$$

Resolution

What about the ingoing solution?

$$\lim_{r\to\infty} \psi \sim e^{-i\omega(t+r_*)}\frac{A}{r}\left(1+\cdots\right).$$

Change to outgoing null coordinate

$$u = t - r_*.$$

Does not blow up, but does oscillate infinite number of times as $r \to \infty$

$$\lim_{r\to\infty} \psi \sim e^{-i\omega u-2i\omega r_*}\frac{A}{r}\left(1+\cdots\right).$$

Similar story for QNMs near the black hole horizon

Ingoing boundary condition near black hole horizon r_+

$$\lim_{r_* \to -\infty} \psi \propto e^{-i\omega(t+r_*)} \left(1 + \cdots\right).$$

Change to ingoing null coordinate

$$v = t + r_*.$$

Ingoing (to horizon) solution regular at horizon

$$\lim_{r \to -\infty} \psi \propto e^{-i\omega v} \left(1 + \cdots\right).$$

Outgoing solution is regular, but oscillates an infinite number of times

$$\lim_{r \to -\infty} \psi \propto e^{-i\omega v + 2i\omega r_*} \left(1 + \cdots\right).$$

Compactified solution

1. Compactify r_*, e.g.
$$r_* = \frac{x}{\sqrt{1-x^2}}.$$

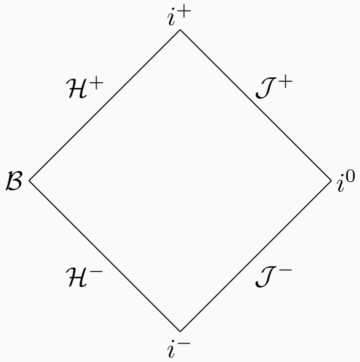
2. Solutions that obey boundary conditions: regular, finite *differentiable* at the boundaries $x = \pm 1$.

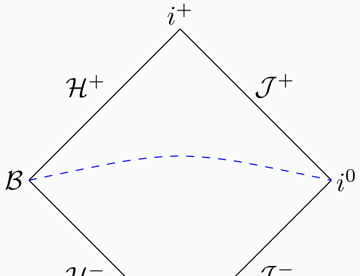
3. Satisfy correct boundary conditions:
$$\lim_{x \to 1} \psi \sim e^{-i\omega u} \frac{A}{r} \left(1 + \cdots\right).$$
$$\lim_{x \to -1} \psi \sim e^{-i\omega v} \left(1 + \cdots\right).$$

4. Don't satisfy correct boundary conditions:
$$\lim_{x \to 1} \psi \sim e^{-i\omega u - 2i\omega r_*} \frac{A}{r} \left(1 + \cdots\right).$$
$$\lim_{x \to -1} \psi \sim e^{-i\omega v + 2i\omega r_*} \left(1 + \cdots\right).$$

Coordinates for Kerr (e.g. Macedo 2019)

$\mathcal{H}^+$: future horizon:
$\{u \to \infty, t, r \text{ finite}\}$
Causally connected to the interior $\mathcal{J}^+$: future null infinity:
$\{v \to \infty, u \text{ finite}\}$
Causally connected to the interior i^+: timelike infinity:
$\{t \to \infty,\ r \text{ finite}\}$
Causally connected to the interior i^0: spatial infinity:
$\{r \to \infty,\ t \text{ finite}\}$
Causally connected to $\mathcal{J}^+$, $\mathcal{J}^-$, i^+, i^- $\mathcal{B}$: bifurcation sphere:
$\{r_* \to -\infty,\ t \text{ finite}\}$
Causally connected to $\mathcal{H}^+$, $\mathcal{H}^-$, i^+, i^- **Boyer-Lindquist** coordinates: constant time

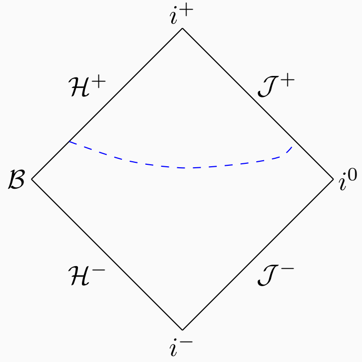




HPHC coordinates for Kerr (e.g. Macedo 2019)

Horizon penetrating, hyperboloidally compactified (**HPHC**) coordinates

1. HPHC coordinates: interpolate between horizon penetrating near $\mathcal{H}^+$, and outgoing null (u) near $\mathcal{J}^+$
2. QNMs are regular in these coordinates (Zenginoğlu 2011, JLR 2022)



HPHC coords + Teukolsky eqn. (Macedo 2019, JLR 2022)

1. Teukolksy equation in Boyer-Lindquist coordinates:

$$\mathcal{T}\Psi_4^{(1)} = 0.$$

2. Transform to ingoing coordinates

$$dv \equiv dt + \frac{2Mr}{\Delta}, \qquad d\phi \equiv d\varphi + \frac{a}{\Delta} dr.$$

3. Rescale $\Psi_4^{(1)}$

$$\psi \equiv \frac{1}{r}\Delta^2 \Psi_4^{(1)}.$$

4. Define "hyperboloidal time variable" τ

$$d\tau \equiv dt + \frac{dh}{dr} dr, \qquad \frac{dh}{dr} \equiv -\left(1 + \frac{4M}{r}\right).$$

5. Radially compactify

$$\rho \equiv \frac{1}{r}.$$

QNMs in HPHC coordinates

1. Decompose ψ

$$\psi = e^{-i\omega\tau + im\phi} R(\rho) S(\theta).$$

2. Teukolsky equation separates,

$$A_1 \frac{d^2 R}{d\rho^2} + B_1 \frac{dR}{d\rho} + C_1 R = \Lambda S,$$
$$A_2 \frac{d^2 S}{d\theta^2} + B_2 \frac{dS}{d\theta} + C_2 S = \Lambda S.$$

3. QNM solutions: find the ω for which there is a **regular** solution R, S to the separated equations.
   - ω: quasinormal mode
   - ψ: quasinormal eigenfunction: regular from $\mathcal{H}^+$ to $\mathcal{J}^+$ on fixed τ hypersufaces

Regular QNM eigenfunctions

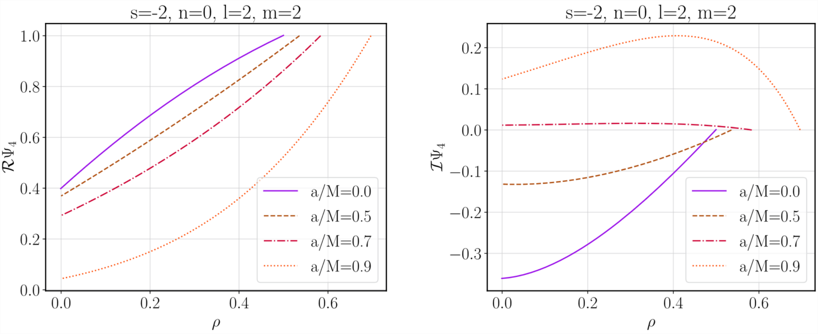
https://github.com/JLRipley314/TeukolskyQNMFunctions.jl

$$A_1 \frac{d^2 R}{d\rho^2} + B_1 \frac{dR}{d\rho} + C_1 R = \Lambda R,$$

$$A_2 \frac{d^2 S}{d\theta^2} + B_2 \frac{dS}{d\theta} + C_2 S = \Lambda S.$$

1. Find ω, R, S that satisfy the simultaneous eigenvalue problem from separation constant Λ
2. Spectral discretiation of S equation in terms of spin-weighted spherical harmonics (Cook+Zalutskiy 2014)
3. Spectral discretiation of R equation in terms of ultraspherical polynomials (Olver + Townsend 2012)
4. Both discretizations: "automatically" impose correct boundary conditions
5. Incorporating improved radial spectral representation from `ApproxFun.jl`: Hengrui Zhu

Regular QNM eigenfunctions



QNMs in HPHC coordinates

`https://github.com/JLRipley314/TeukolskyQNMFunctions.jl`

Collaborator: Hengrui Zhu (graduate student, Princeton University)

More input/pull requests welcome! (improve algorithms, etc)

Application: measuring QNMs

With HPHC coordinates, can for the first time study *exact* (superpositions of) QNM initial data

How well (in principle) can you measure the QNM content of an arbitrary ringdown signal?

Challenges in Quasinormal Mode Extraction: Perspectives from Numerical solutions to the Teukolsky Equation

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(Dated: October 9, 2023)

The problem

1. Quasinormal modes are not complete
2. Linear theory: there are *non-mode* solutions to the Teukolsky equation
3. There are *nonlinearities* in solutions to full Einstein equations
4. Could non-mode contributions bias your measurement of the quasinormal modes?

What modes are present in the ringdown from binary coalescence?

Testing the no-hair theorem with GW150914

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[4]*Department of Physics and Astronomy, Stony Brook University, Stony Brook NY 11794, USA*
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(Dated: August 12, 2019)

We analyze gravitational-wave data from the first LIGO detection of a binary black-hole merger (GW150914) in search of the ringdown of the remnant black hole. Using observations beginning at the peak of the signal, we find evidence of the fundamental quasinormal mode and at least one overtone, both associated with the dominant angular mode ($\ell = m = 2$), with 3.6σ confidence. A ringdown model including overtones allows us to measure the final mass and spin magnitude of the remnant exclusively from postinspiral data, obtaining an estimate in agreement with the values inferred from the full signal. The mass and spin values we measure from the ringdown agree with

Analysis of Ringdown Overtones in GW150914

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Our setup

1. What modes can we measure in an *idealized (numerical) setting*?
2. QNM initial data in HPHC coordinates

   https://github.com/JLRipley314/TeukolskyQNMFunctions.jl

3. Linear *evolution* code in HPHC coordinates

   https://github.com/JLRipley314/TeukEvolution.jl

4. *Fixed* black hole background (M, a do not change)
5. Numerical "laboratory"
   - Pure QNM initial data
   - Non-QNM initial data

Time evolution code

1. https://github.com/JLRipley314/TeukEvolution.jl
2. Solve Teukolsky equation in time domain

$$\mathcal{T}_{-2}\Psi_4^{(1)} = 0.$$

3. Decompose $\Psi_4^{(1)} e^{im\phi}$: $2+1$ evolution for each m-mode
4. Spectral decomposition in spin-weighted spherical harmonics in the angular directions
5. Finite difference in radial direction
6. Method of times (4^{th} order Runge-Kutta) in time direction
7. Incorporating exact QNM initial data from `TeukolskyQNMFunctions.jl`: Hengrui Zhu

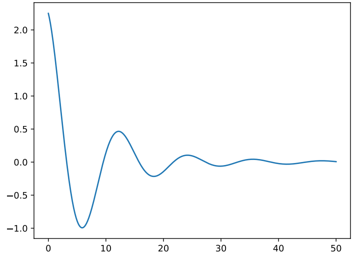
Fitting quasinormal modes: traditional approach

- Find best best fit to extrapolated waveform at future null infinity

$$\Psi_4^{(1)} \sim \sum_{\ell,m,n} A_{\ell,m,n} e^{-i\omega_{\ell,m,n} t}.$$

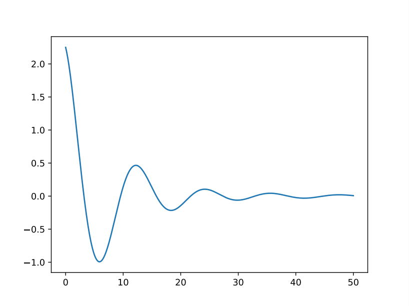
- Minimize quadratic residual (least squares)

$$\min_{A_{\ell,m,n}} \left\{ \sum_{\ell,m,n} \left(A_{\ell,m,n} e^{-i\omega_{\ell,m,n} t} - s\left(t\right)\right)^2 \right\}$$



Taking only information at future null infinity

- Do not look at radial information: losing valuable signal-to-noise in fit
- Could a signal "look like" at QNM at a fixed radius, but differ as the radius changes?
- HPHC coordinates: straightforward way to incorporate radial information in the QNM fit



Bilinear form (least squares) QNM fit

1. Define the mode (fixed m mode)

$$\Phi_{[p],\ell,n}\left(t_0,\rho_j,\theta_k\right) \equiv e^{-i\omega_{[p],\ell,n}t_i} R_{[p],\ell,n}\left(\rho_j\right) S\left(a\omega_{[p],\ell,n},\theta_k\right).$$

2. QNM fit

$$\Psi_4\left(t_i,\rho_j,\theta_k\right) = \sum_{p,n,\ell} A_{[p],n,\ell}\Phi_{[p],\ell,n}\left(t_i,\rho_j,\theta_k\right),$$

$$\partial_t\Psi_4\left(t_i,\rho_j,\theta_k\right) = \sum_{p,n,\ell} A_{[p],n,\ell}\left(-i\omega_{[p],\ell,n}\right)\Phi_{[p],\ell,n}\left(t_i,\rho_j,\theta_k\right).$$

3. Write as matrix equation (I: (i,j,k); J: $[p],\ell,n$)

$$\begin{pmatrix}\Psi_I\\ \partial_t\Psi_I\end{pmatrix} = \begin{pmatrix}M_{IJ} & 0\\ 0 & -i\omega_J M_{IJ}\end{pmatrix} A_J \equiv N_{IJ}A_J.$$

4. Find A_J: matrix inversion

$$A_J = N_{IJ}^{-1}\begin{pmatrix}\Psi_I\\ \partial_t\Psi_I\end{pmatrix}$$

Bilinear form (least squares) QNM fit

1. Matrix equation (I: (i, j, k); J: $[p], \ell, n$)

$$A_J = N_{IJ}^{-1} \begin{pmatrix} \Psi_I \\ \partial_t \Psi_I \end{pmatrix}$$

2. Time-only fit: fix j, k

$$A_J = N_{IJ}^{-1} \begin{pmatrix} \Psi_I \\ \partial_t \Psi_I \end{pmatrix}$$

3. Fixed time fit: fix i
4. Whole domain fit: sum over i, j, k

Bilinear form (least squares) QNM fit

1. Matrix equation (I: (i, j, k); J: $[p], \ell, n$)

$$A_J = N_{IJ}^{-1} \begin{pmatrix} \Psi_I \\ \partial_t \Psi_I \end{pmatrix}$$

2. Linear regression $\sim$ minimize quadratic residual

$$r = \sum_{IJ} \left(\Psi_I - \sum_J A_J \Phi_{IJ} \right)^2 + \sum_{IJ} \left(\partial_t \Psi_I + \sum_J i\omega_J A_J \Phi_{IJ} \right)^2 .$$

WARNING: the QNMs are not complete.

This method *will* fit modes to non-mode components to solution.

NOTE: we do something slightly different in **2309.13204** (projection along spherical harmonics).

“Stability” of a fit (London+ 2014, London 2018, Baibhav + 2023)

1. How does the fit for A_J change with time?

$$A_J = N_{IJ}^{-1} \begin{pmatrix} \Psi_I \\ \partial_t \Psi_I \end{pmatrix}$$

2. Fixed time fit: $\left|A_J\left(t_i\right)\right| e^{\mathcal{I}\omega_J t_i}$
3. Time interval fit: $\left|A_J\left(t_{i_1}, t_{i_2}\right)\right| e^{\mathcal{I}\omega_J t_i}$
4. Constant A_J (with time): have measured a QNM
5. Varying A_J: fitting a mode to non-mode component to the solution

Numerical experiment: pure QNM initial data

Initial data: pure sum of QNMs

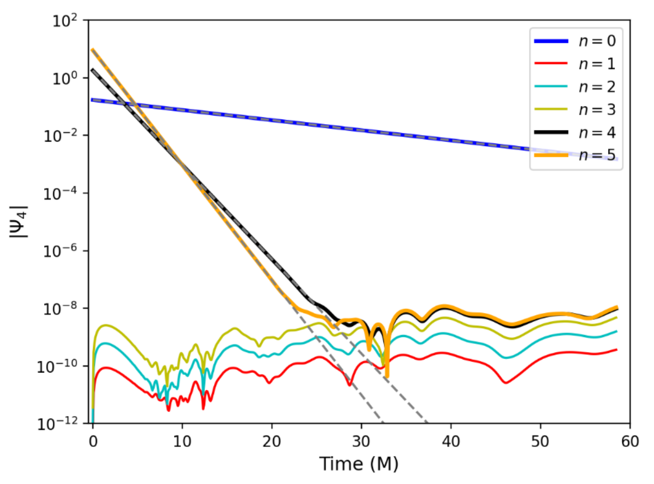
$$\Psi_4^{(1)}\Big|_{\tau=0} = \sum_{[p],\ell,m,n} A_{[p],\ell,m,n} R_{[p],\ell,m,n}\left(\rho\right) S_{\ell,m}\left(a\omega_{[p],\ell,m,n},\theta\right) e^{im\phi},$$

$$\partial_t \Psi_4^{(1)}\Big|_{\tau=0} = \sum_{[p],\ell,m,n} -i\omega_{[p],\ell,m,n} A_{[p],\ell,m,n} R_{[p],\ell,m,n}\left(\rho\right) S_{\ell,m}\left(a\omega_{[p],\ell,m,n},\theta\right) e^{im\phi}.$$

No non-mode contributions to signal.

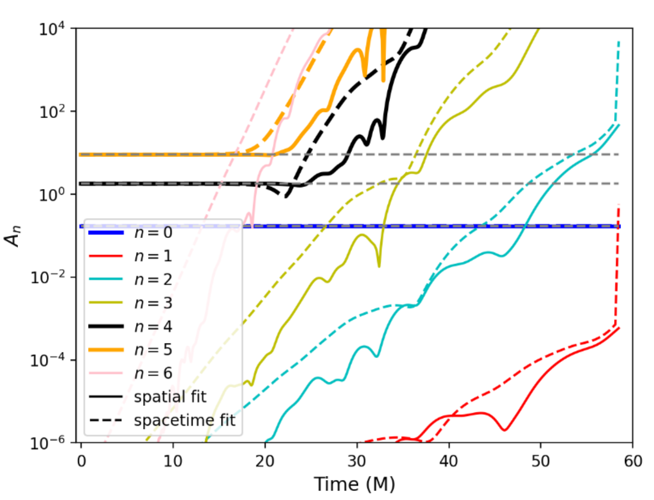
Numerical experiment: pure QNM initial data

Initial data: pure sum of QNMs



“Stability” of fit

Rescale each mode by expected decay $|A|e^{\mathcal{I}\omega t}$



Modes initially not in initial data are *always* unstable

Numerical experiment: pure QNM initial data

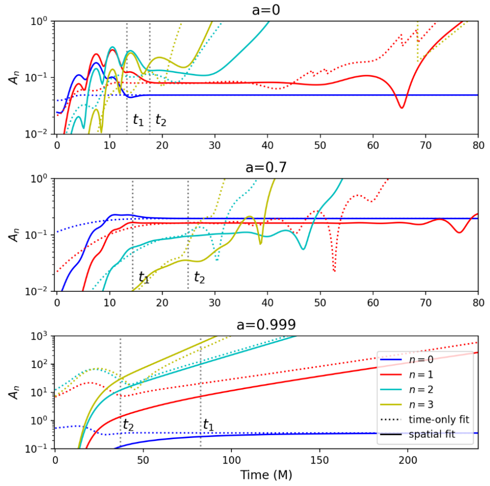
Initial data: Gaussian-like pulse

$$\left.\Psi_4^{(1)}\right|_{\tau=0} = \exp\left(-\left(\frac{\rho^{-1} - r_0}{w}\right)^2\right) {}_{-2}Y_\ell\left(\theta\right),$$
$$\left.\partial_t \Psi_4^{(1)}\right|_{\tau=0} = -\frac{\rho^2}{2+4\rho}\partial_\rho\left(\left.\Psi_4^{(1)}\right|_{\tau=0}\right).$$

Non-mode contributions to initial data.

“Stability” of fit: spatial fitting vs time-only fitting

Rescale each mode by expected decay



Spatial fitting at a *fixed* time more stable that fitting time only over a finite time interval.

Closer look at time-only fit

1. Time-only fit appears to be less stable (does not give consistent fit except for 1-2 modes)
2. To evaluate accuracy of time-only fit
   2.1 Compute time-only fit at several different, *finite* radii and at future null infinity
   2.2 Family of amplitudes $\rightarrow A(\rho_i)$
3. *If* fitting a QNM, then expect

$$A(\rho_i) = AR(\rho)$$

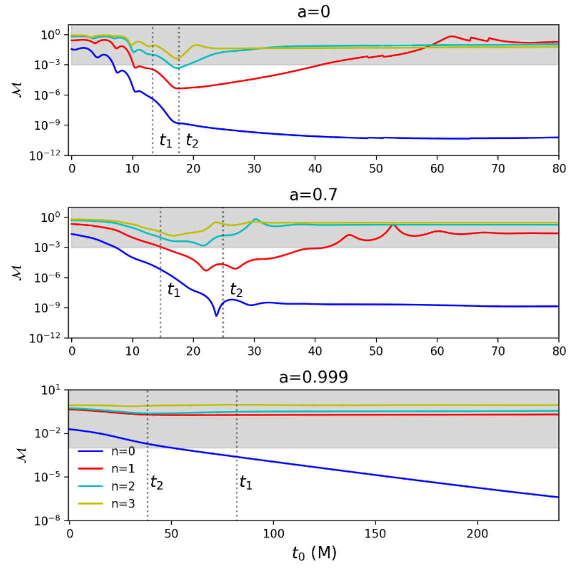
4. Judge how close this is to being true by calculating

$$\mathcal{M} \equiv 1 - \left| \frac{\langle A(\rho), R(\rho) \rangle}{\langle A(\rho), A(\rho) \rangle \langle R(\rho), R(\rho) \rangle} \right| .$$

$\langle a, b \rangle \equiv \int d\rho \bar{a}(\rho) b(\rho)$

Mismatch for time-like fits

Mismatch as a function of the start time

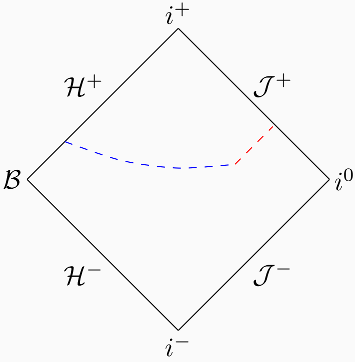


Conclusions from experiments

1. Considered an *idealized* scenario: linearized evolution about Kerr black hole background
2. With pure QNM initial data, can extract modes easily
3. With more complicated initial data, difficulties extracting more than **1-2 modes** when using a time-only fit (no radial information)
4. Including radial information greatly stabalizes QNM fitting

Potential extension to nonlinear codes

1. Do not consider entire domain–consider radial information in the *radiation zone*, where gravitational fields are weak
2. Does not need to be hyperboloidal–characteristic coordinates would also work (e.g. Cauchy Characteristic Extraction codes; Ma+ 2023)



Conclusion

1. QNMs regular, well behaved functions in HPHC coordinates
2. Can study *exact* QNM initial data in HPHC coordinates
3. Can use *radial* QNM information in HPHC coordinates $\rightarrow$ more robust QNM fitting
4. Future work
   - Radial fits in nonlinear codes
   - Exact QNM initial data for second order perturbative codes
5. More detals
   - Zhu+ **2309.13204**, JLR **2202.03837**
   - Evolution code https://github.com/JLRipley314/TeukEvolution.jl
   - QNM code https://github.com/JLRipley314/TeukolskyQNMFunctions.jl

Backup slides

Newman-Penrose formalism

Complex, null tetrad

$$l_\mu, n_\mu, m_\mu, \bar{m}_\mu$$

Metric→tetrad

$$g_{\mu\nu} = 2l_{(\mu}n_{\nu)} - 2m_{(\mu}\bar{m}_{\nu)}$$

Derivatives→ directional derivatives

$$D \equiv l^\mu \nabla_\mu, \qquad \Delta \equiv n^\mu \nabla_\mu, \qquad \delta \equiv m^\mu \nabla_\mu$$

Christoffel symbols→Ricci rotation coefficients; e.g.

$$\sigma \equiv m^\mu m^\nu \nabla_\mu l_\nu$$

Curvature tensors→Curvature scalars; e.g.

$$\Psi_4 \equiv C_{\mu\nu\alpha\beta} n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta$$

NP Equations of motion

Projections of higher derivative equations

Projections of $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$

Projections of $\nabla^{\mu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0$

Projections of $l^{\mu} \nabla_{\mu} n^{\nu} - n^{\mu} \nabla_{\mu} l^{\nu} = \cdots$

1. **Disadvantage:** Lots of variables and complicated equations
2. **Advantage(s):** Have **ten** degrees of freedom (not just four) to choose
    - Four gauge (coordinate) degrees of freedom to set x^{μ}
    - Six tetrad degrees of freedom $l_{\mu}, n_{\mu}, m_{\mu}, \bar{m}_{\mu}$
3. With a good choice of tetrad, the linearized Einstein equations take a simple form around Kerr black holes