Computing the second order gravitational perturbation of Kerr black holes[1]

Friday GR seminar

Justin Ripley

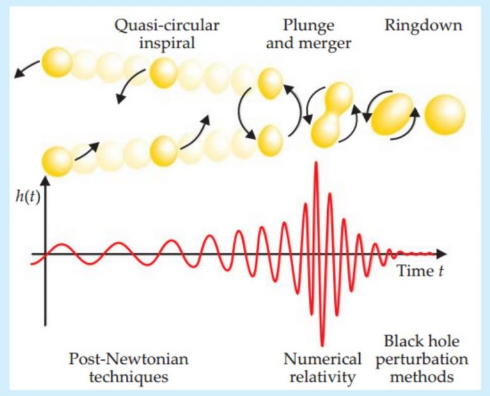
with Nicholas Loutrel, Elena Giorgi, and Frans Pretorius

DAMTP, Cambridge University

November 12, 2020

[1] arXiv:2008.11770,arXiv:2010.00162

Question:[2]



How well does linear perturbation theory describe the geometry of the deformed black hole formed after the merger of two black holes (or another compact object with a black hole)?

[2] *Image: Baumgarte and Shapiro, Numerical Relativity: Solving Einstein's Equations on the Computer*

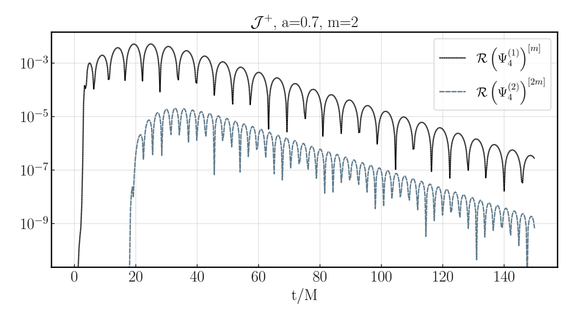
Black hole ringdown: higher order effects?

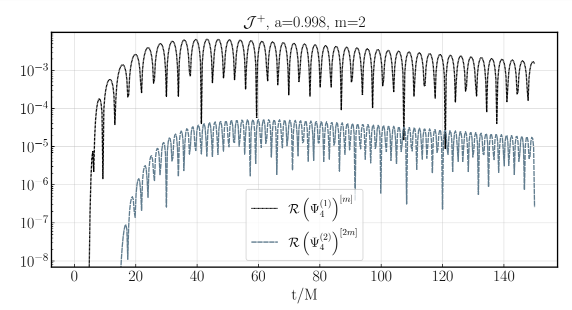
- Ringdown modeled as sum of quasinormal modes

$$h(t) = \mathcal{R} \sum_{n,l,m} A_{n,l,m} e^{-i\omega_{nlm} t}.$$

- Modes determined solely by final black hole spin and mass
- Need to measure higher modes for no hair tests, e.g. $\omega_{n,l,m} = \omega_{0,2,2}, \omega_{0,4,4}$
- Does linear perturbation theory model excitation of higher order modes, or does nonlinear mode coupling significantly contribute to amplitude of higher order modes?

Goal: understand these plots, and how we computed them





Outline

Gravitational perturbation theory: the formalism we use

Numerical implementation

Results

Future directions/work in progress

Gravitational perturbation theory

$$G_{\mu\nu}[g] = 0.$$

Expand around background spacetime ($\epsilon \ll 1$)

$$\begin{aligned} g_{\mu\nu} =& g^{(0)}_{\mu\nu} + \epsilon g^{(1)}_{\mu\nu} + \epsilon^2 g^{(2)}_{\mu\nu} + \cdots \\ G_{\mu\nu} =& \epsilon G^{(0)}_{\mu\nu}\left[g^{(1)}\right] + \epsilon^2 \left(G^{(1)}_{\mu\nu}\left[g^{(1)}\right] + G^{(0)}_{\mu\nu}\left[g^{(2)}\right]\right) + \cdots \end{aligned}$$

Order by order

$$\begin{aligned} G^{(0)}_{\mu\nu}\left[g^{(1)}\right] =& 0, \\ G^{(0)}_{\mu\nu}\left[g^{(2)}\right] =& -G^{(1)}_{\mu\nu}\left[g^{(1)}\right] \\ \cdots \end{aligned}$$

Newman-Penrose formalism: curvature perturbation

Complex, null tetrad

$$l_\mu, n_\mu, m_\mu, \bar{m}_\mu$$

Metric→tetrad

$$g_{\mu\nu} = 2l_{(\mu}n_{\nu)} - 2m_{(\mu}\bar{m}_{\nu)}$$

Derivatives→ directional derivatives

$$D \equiv l^\mu \nabla_\mu, \qquad \Delta \equiv n^\mu \nabla_\mu, \qquad \delta \equiv m^\mu \nabla_\mu$$

Christoffel symbols→Ricci rotation coefficients; e.g.

$$\sigma \equiv m^\mu m^\nu \nabla_\mu l_\nu$$

Curvature tensors→Curvature scalars; e.g.

$$\Psi_4 \equiv C_{\mu\nu\alpha\beta} n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta$$

Newman-Penrose formalism: curvature perturbation

Newman-Penrose scalar ($C_{\mu\nu\alpha\beta}$ Weyl tensor):

$$\Psi_4 \equiv C_{\mu\nu\alpha\beta} n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta .$$

$$\begin{aligned} \Psi_4^{(Kerr)} =& 0, \\ \Psi_4 =& \Psi_4^{(Kerr)} + \epsilon \Psi_4^{(1)} + \epsilon^2 \Psi_4^{(2)} + \mathcal{O}\left(\epsilon^3\right). \end{aligned}$$

Order by order solve the **Teukolsky equation**

$$\begin{aligned} \mathcal{T}_{-2} \Psi_4^{(1)} =& 0 \\ \mathcal{T}_{-2} \Psi_4^{(1)} =& \mathcal{S}\left[g_{\mu\nu}^{(1)}\right] \\ & \cdots \end{aligned}$$

Steps

1. Linear gravitational perturbation

$$\mathcal{T}_{-2}\Psi_4^{(1)} = 0.$$

2. Metric reconstruction

$$\Psi_4^{(1)} \to g_{\mu\nu}^{(1)}.$$

3. Second order gravitational perturbation

$$\mathcal{T}_{-2}\Psi_4^{(2)} = \mathcal{S}\left[g_{\mu\nu}^{(1)}\right].$$

4. Relate $\Psi_4^{(1)}$ and $\Psi_4^{(2)}$ to gravitational wave radiation at future null infinity

Equations of motion for the linear curvature perturbation: Teukolsky equation[3]

PERTURBATIONS OF A ROTATING BLACK HOLE. I. FUNDAMENTAL EQUATIONS FOR GRAVITATIONAL, ELECTROMAGNETIC, AND NEUTRINO-FIELD PERTURBATIONS*

SAUL A. TEUKOLSKY†
California Institute of Technology, Pasadena
Received 1973 April 12

field ($s = \pm 1$, derived in § III), or a gravitational perturbation ($s = \pm 2$, derived in § II):

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$
$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T\,. \quad (4.7)$$

Here s is a parameter called the "spin weight" of the field. Table 1 specifies the field

Schematically:

$$\mathcal{T}_{-2}\Psi_4^{(1)} = 0.$$

[3] Teukolsky, ApJ Vol. 185, pp 635-648 (1973)

Equation of motion for second order perturbation

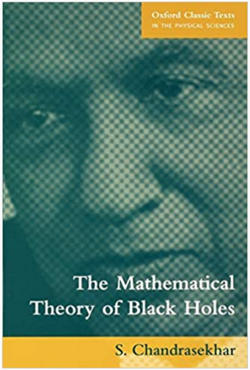
Equation of motion[4] for $\Psi_4^{(2)}$:

$$\mathcal{T}_{-2}\Psi_4^{(2)} = \mathcal{S}\left[g_{\mu\nu}^{(1)}\right].$$

We need to *reconstruct* $g_{\mu\nu}^{(1)}$ from $\Psi_4^{(1)}$.

[4] Campanelli and Lousto, Phys.Rev.D 59 (1999) 124022

Direct metric reconstruction



We can use Bianchi and Ricci identities to obtain transport equations for metric components[5]

[5] Chandrasekhar, 1992; Andersson, Bäckdahl, Blue, Ma, 1903.03859

Direct metric reconstruction

Example equation

$$0 = -\left(\Delta + \mu + \bar{\mu}\right)\lambda^{(1)} - \Psi_4^{(1)}$$

This is a transport equation for $\lambda^{(1)}$, as

$$\Delta = \left(2 + \frac{4MR}{L^2}\right)\partial_T + \frac{R^2}{L^2}\partial_R$$

Need to specify Cauchy data for $\lambda^{(1)}$

Direct metric reconstruction

$$
\begin{array}{ccccccc}
\mathcal{T}_{-2}\left[\Psi_4^{(1)}\right] = 0 & \Longrightarrow \Psi_4^{(1)} & \Longrightarrow & \lambda^{(1)} & \Longrightarrow & h_{mm} \\
 & \Downarrow & & & & \\
 & \Psi_3^{(1)} & \Longrightarrow & \pi^{(1)} & \Longrightarrow & h_{lm} \\
 & \Downarrow & & & & \\
 & \Psi_2^{(1)} & \Longrightarrow & h_{ll} & &
\end{array}
$$

$h_{mm} \equiv g^{(1)}_{\mu\nu} m^\mu m^\nu$, etc.

Relating curvature perturbation to metric perturbation at future null infinity

$$\lim_{r\to\infty}\Psi_4^{(1)} = -\,\partial_t^2 g_+^{(1)} + i\partial_t^2 g_\times^{(1)}.$$

In an **asymptotically flat gauge**, for **outgoing waves** at future null infinity, we have[6]

$$\lim_{r\to\infty}\Psi_4^{(2)} = -\,\partial_t^2 \delta g_+^{(2)} + i\partial_t^2 \delta g_\times^{(2)},$$

We use outgoing radiation gauge (asymptotically flat), and our initial data sets no ingoing radiation at future null infinity

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[6] Campanelli and Lousto, Phys.Rev.D 59 (1999) 124022

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Steps

1. Linear gravitational perturbation: **solve the Teukolsky equation**

$$\mathcal{T}_{-2}\Psi_4^{(1)} = 0.$$

2. Metric reconstruction: **direct reconstruction by solving a nested set of linear transport equations**

$$\Psi_4^{(1)} \to g_{\mu\nu}^{(1)}.$$

3. Second order gravitational perturbation: **solve the Teukolsky equation**

$$\mathcal{T}_{-2}\Psi_4^{(2)} = \mathcal{S}\left[g_{\mu\nu}^{(1)}\right].$$

4. Relate $\Psi_4^{(1)}$ and $\Psi_4^{(2)}$ to gravitational wave radiation at future null infinity: **read off values of Ψ_4 at future null infinity**

Numerical setup: $1+2$ dimensional evolution

Kerr spacetime: axially symmetric:

$$f(T,R,\theta,\phi)=\sum_m f^{[m]}(T,R,\theta)e^{im\phi}.$$

Solve in the time domain

Form of mode mixing

Second order:

$$\mathcal{T}_{-2}\Psi_4^{(2)} = \mathcal{S}\left[g_{\mu\nu}^{(1)}\right] " = " \left(\Psi_4^{(1)}\right)^2, \Psi_4^{(1)}\bar{\Psi}_4^{(1)}$$

Nonlinear mode mixing:

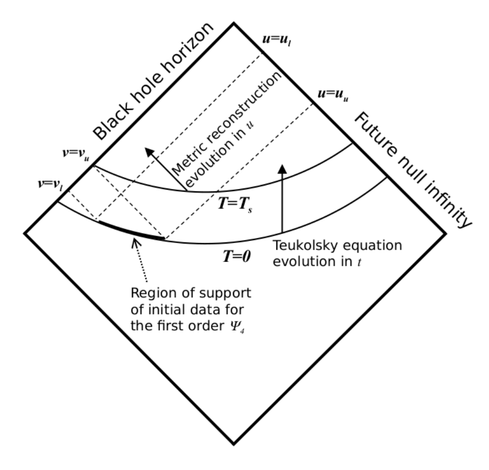
$$\Psi_4^{(1)[\pm 2]} e^{2i\phi} \to_{\mathcal{S}} \left\{\Psi_4^{(2)[\pm 4]} e^{4i\phi}, \Psi_4^{(2)[0]} e^{0i\phi}\right\} \tag{1}$$

More generally

$$\{\pm m_1, \pm m_2\} \to \{\pm 2m_1, \pm 2m_2, \pm (m_1 \pm m_2)\} \tag{2}$$

Coordinates and initial data

Pseudospectral code that solves the Teukolsky equation in horizon penetrating, hyperboloidally compactified coordinates



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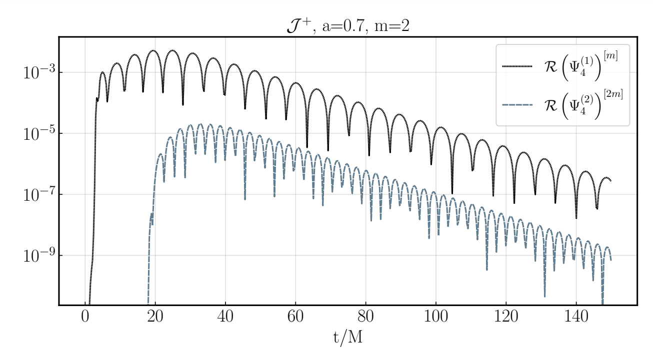
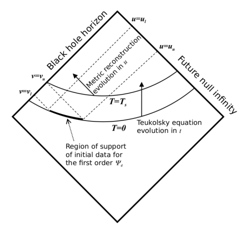
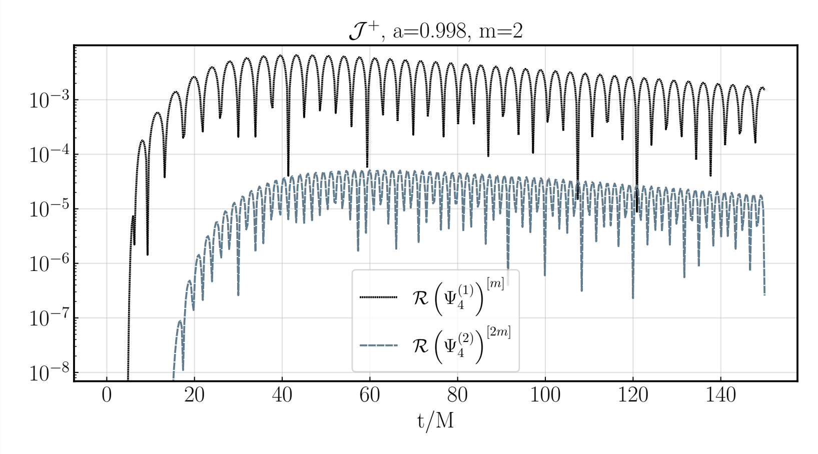
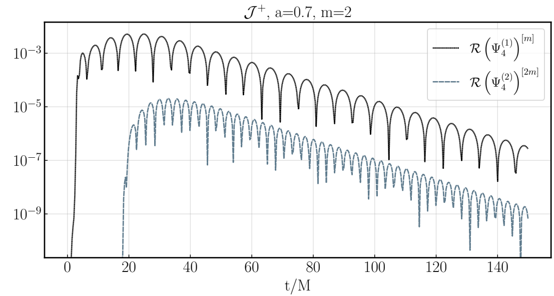


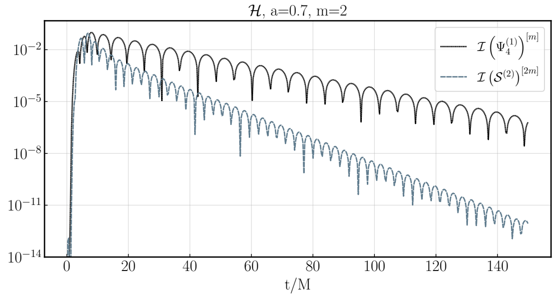


Second order evolution



Second order source vs. second order curvature





Decay of source vs. decay of curvature

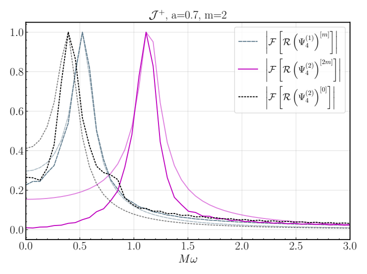
- The second order curvature perturbation does *not* decay as $\Psi_4^{(2)} \propto \left(\Psi_4^{(1)}\right)^2$ at late times
- Instead, $\mathcal{S} \propto \left(\Psi_4^{(1)}\right)^2$
- Homogeneous vs. particular solution

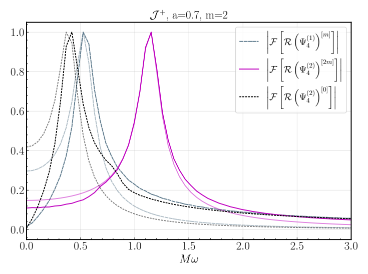
$$\mathcal{T}_{-2}\Psi_4^{(2)} = \mathcal{S}\left[g_{\mu\nu}^{(1)}\right]$$

- In the ordinary perturbation theory approach we use, at late times second and first order perturbation decay roughly at the same rate: quasinormal mode of ω_{nlm} (black hole spin $a = 0.7$):

$$\omega_{022}/M \approx 0.5 - 0.08i$$
$$\omega_{044}/M \approx 1.1 - 0.08i$$

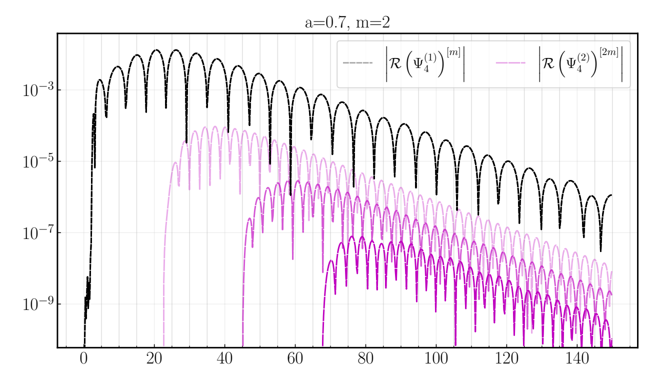
Fourier transform of signal

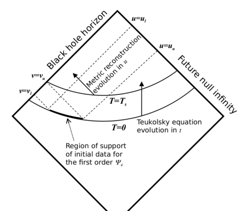




- Left: Fourier transform at early times
- Right: Fourier transform at late times
- Late times: dynamics determined by slowest decaying quasinormal mode

Relative size of second order perturbation as a function of start time





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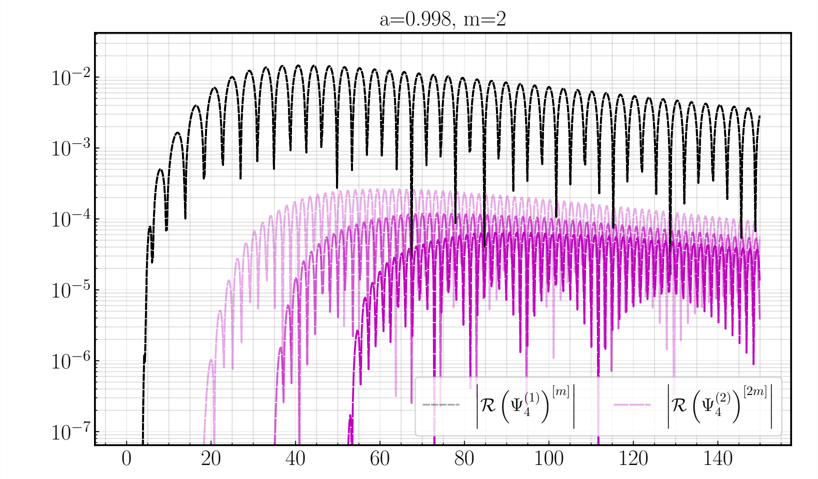
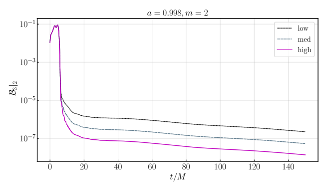
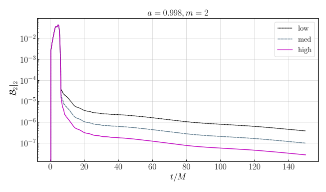
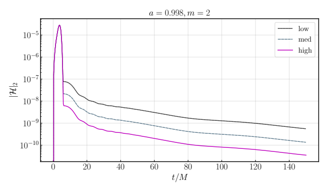


Convergence of metric reconstruction in code







Convergence of independent residuals: two Bianchi identities, and testing $h_{//}$ is a real variable

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Future directions

- Explore nonlinear black hole physics; e.g. “turbulent” gravitational wave interactions
- Map out energy cascade between modes
- Astrophysically realistic initial data
- Metric reconstruction with matter fields (application: self force)

Energy cascade of modes and "turbulent" gravitational wave interactions

- Consider near extremal Kerr black holes: $a \to M$.
- In near extremal limit there is a family of "zero-damped modes"[7], whose decay timescale goes as $T/M \sim (1 - a/M)^{-1/2}$
- At extremal limit have terms that do not decay at all on the horizon: Aretakis instability[8]
- Zero-damped modes decay most slowly near the black hole horizon: gradients may grow
- Growing gradients: nonlinear effects may be important
- Heuristic arguments: nonlinear gravitational wave interactions may have properties similar to homogeneous two-dimensional fluid turbulence[9]

[7] e.g. Hod Phys.Rev.D 78 (2008); Yang et. al., Phys.Rev.D 88 (2013)
[8] Adv.Theor.Math.Phys. 19 (2015) 507-530
[9] Yang et. al., Phys.Rev.Lett, 114 (2015)

Energy cascade of modes and "turbulent" gravitational wave interactions

- Challenge: how to measure "turbulent" gravitational wave interactions in a gauge and coordinate invariant way
- Challenge: relate Weyl scalar Ψ_4 to some fluid variable to make connection to Turbulence
- Consider: spectrum of energy flux radiated through future null infinity

$$\frac{dE}{du} = \lim_{r\to\infty} \frac{r^2}{4\pi} \int_{\mathbb{S}^2} d\Omega \left| \int_{-\infty}^{u} d\tilde{u} \Psi_4 \right|^2 .$$

Energy cascade of modes and "turbulent" gravitational wave interactions

- Challenge: how to measure "turbulent" gravitational wave interactions using ordinary perturbation theory
- Turbulence signaled by secular growth in perturbation theory

$$\Psi_4^{(2)} = (\psi_0 + T\psi_1 + \cdots)$$

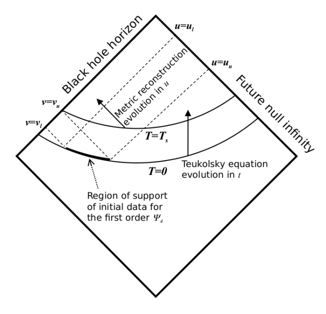
- What is the best way to resum secular growth(dynamical renormalization group, etc), to obtain late time dynamical behavior for near-extremal black holes?

Astrophysically realistic initial data

Ringdown modeled as sum of quasinormal modes

$$h(t) = \mathcal{R} \sum_{n,l,m} A_{n,l,m} e^{-i\omega_{nlm} t}.$$

What are realistic values of the excitation coefficients $A_{n,l,m}$?



Metric reconstruction with matter fields: self-force[10]

Self force problem:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

where $T_{\mu\nu}$ is the stress-energy tensor of a point particle

[10] e.g. Barack and Pound, Rept. Prog. Phys. 82 (2019)

Metric reconstruction with matter fields: self-force

Expand

$$
\begin{aligned}
g_{\mu\nu} =& g^{(0)}_{\mu\nu} + \epsilon g^{(1)}_{\mu\nu} + \epsilon^2 g^{(2)}_{\mu\nu} + \cdots \\
T_{\mu\nu} =& + \epsilon T^{(1)}_{\mu\nu} + \epsilon^2 T^{(2)}_{\mu\nu} + \cdots \\
G_{\mu\nu} =& \epsilon G^{(0)}_{\mu\nu}\left[g^{(1)}\right] + \epsilon^2 \left(G^{(1)}_{\mu\nu}\left[g^{(1)}\right] + G^{(0)}_{\mu\nu}\left[g^{(2)}\right]\right) + \cdots
\end{aligned}
$$

Order by order:

$$
\begin{aligned}
G^{(0)}_{\mu\nu}\left[g^{(1)}\right] =& T^{(1)}_{\mu\nu}, \\
G^{(0)}_{\mu\nu}\left[g^{(2)}\right] =& T^{(2)}_{\mu\nu} - G^{(1)}_{\mu\nu}\left[g^{(1)}\right] \\
& \cdots
\end{aligned}
$$

Metric reconstruction with matter fields: self-force

- Order by order in NP formalism

$$\mathcal{T}_{-2}\Psi_4^{(1)} = T_4^{(1)},$$
$$\mathcal{T}_{-2}\Psi_4^{(2)} = T_4^{(2)} + \mathcal{S}_4$$
$$\cdots$$

- Technical challenge: metric reconstruction $\Psi_4^{(1)} \to g_{\mu\nu}^{(1)}$ with a source term
- We use outgoing radiation gauge, which one cannot use when there are source terms in the linear equations of motion
- Potential go-around: Green, Hollands, Zimmerman, Class. Quant. Grav. 37, (2020), or work instead in an outgoing Bondi gauge

Conclusion[11]

- How well does linear perturbation theory describe the ringdown of a Kerr black hole?
- Computation of second order curvature perturbation of a Kerr black hole: $\Psi_4^{(2)}$
- Direct metric reconstruction of metric from linear curvature perturbation: $\Psi_4^{(1)} \to g_{\mu\nu}^{(1)}$
- Future directions
  - Astrophysically realistic initial data
  - Direct metric reconstruction with matter fields
  - Examine late time nonlinear behavior of nearly-extremal Kerr black holes: do gravitational waves undergo turbulent energy cascades?

[11] arXiv:2008.11770,arXiv:2010.00162

Recent work related to nonlinear perturbations of Kerr

- Using Hertz potentials:
  - Green, Hollands Zimmerman, Class. Quant. Grav. 37 (2020) 075001, arXiv:1908.09095
- Using "Kerrness" measure with nonlinear simulations:
  - Bhagwat et. al. Phys. Rev. D97 (2018) 10, 104065, arXiv:1711.00926
  - Okounkova, arXiv:2004.00671